Recent algorithmic results on equitable coloring

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Equitable Coloring

Equitable (Vertex) Coloring

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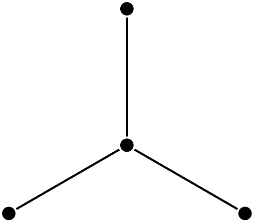

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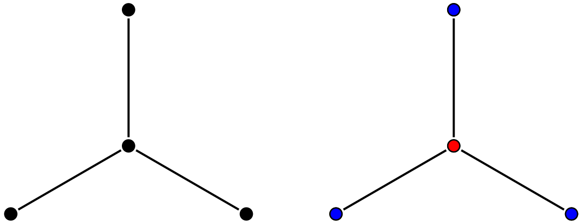

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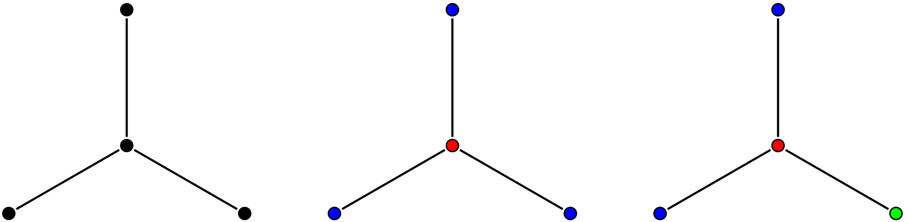

Some important stuff I

Equitable chromatic number

The smallest integer k such that G is equitably k-colorable is the *equitable chromatic number* $\chi_{=}(G)$.

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Equitable Coloring Conjecture

For every connected graph G which is neither a complete graph nor an odd-hole, $\chi_=(G) \leq \Delta(G)$.

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Any graph G is equitably k-colorable if $k \geq \Delta(G) + 1$. Equivalently, $\chi^*_=(G) \leq \Delta(G) + 1$.

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For every connected graph G which is not a complete graph, an odd-hole nor $K_{2n+1,2n+1}$, for any $n \geq 1$, $\chi^*_=(G) \leq \Delta(G)$ holds.

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Ko-Wei Lih. "Equitable coloring of graphs". In: *Handbook of combinatorial optimization*. Springer, 2013, pp. 1199–1248

The story so far...

Class	Complexity
Trees	P
Forests	P
Bipartite	NP-complete, even if $k = 3$
Co-bipartite	P
Cographs	NP-complete, P for each fixed k
Bounded Treewidth	P
Chordal	NP-complete
Block	?
Split	P
Unipolar	?
Interval	NP-complete
Co-interval	P

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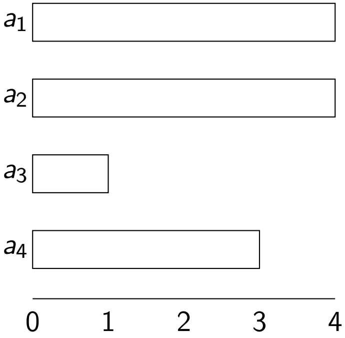

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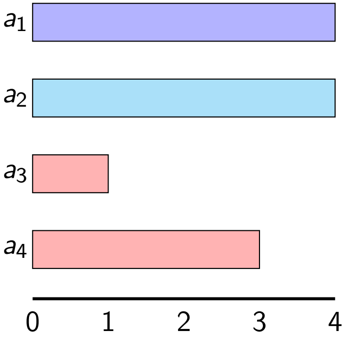

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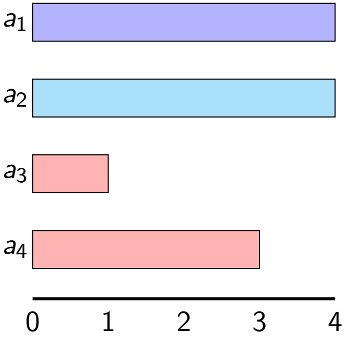

- For each item of A, build a gadget with some **key** vertices.
- All key vertices must have the same color.
- Key vertices with color $i \rightarrow$ item in i-th bin.

Block graphs

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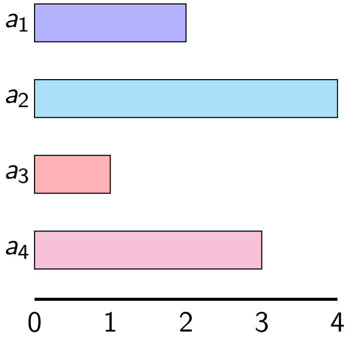

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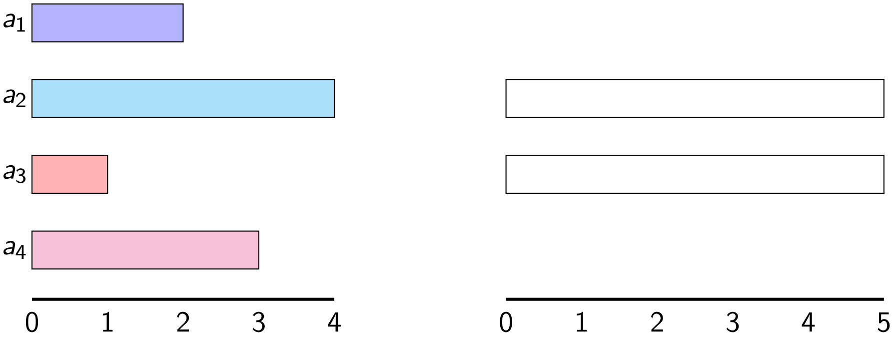

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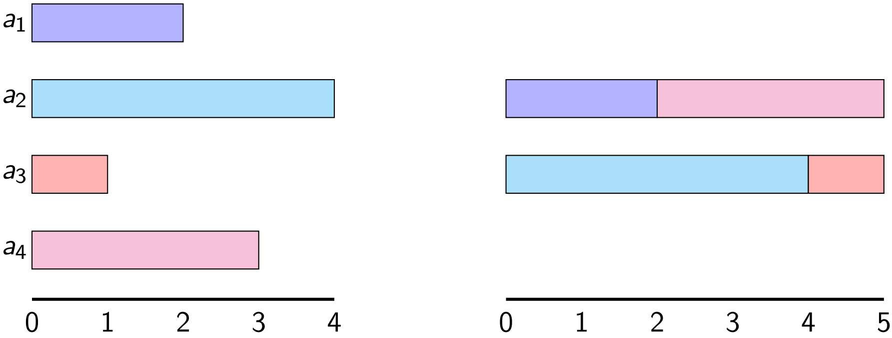

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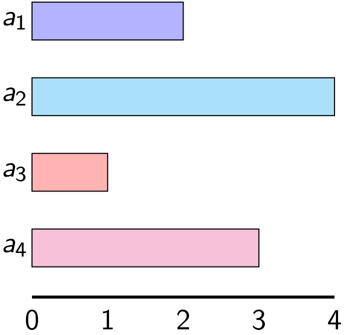

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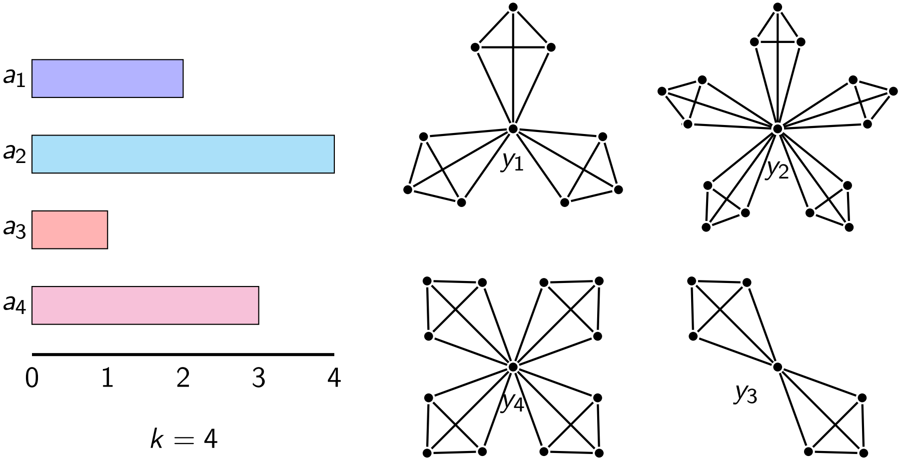

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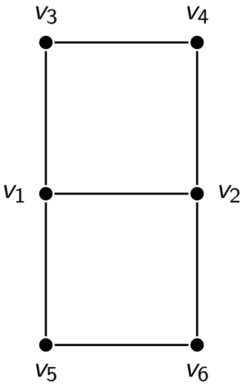

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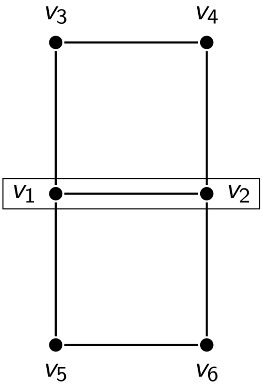
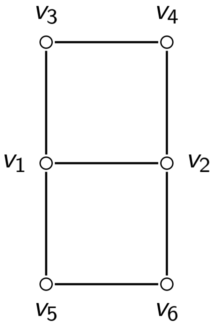


A max-flow based algorithm



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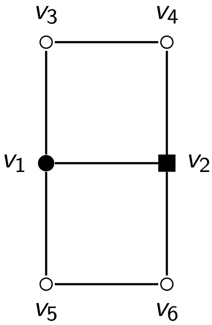

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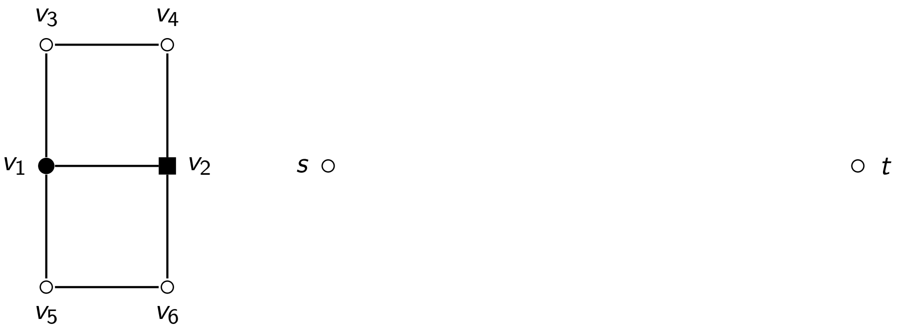
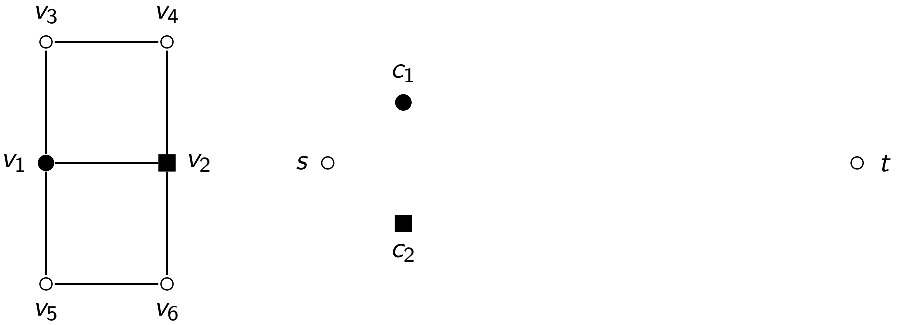


Vertices

Source s, sink t,

A max-flow based algorithm



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Source s, sink t, for each color i, c_i,

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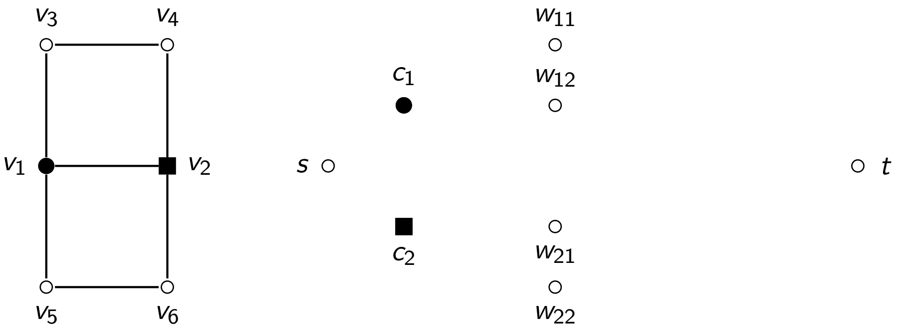

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Source s, sink t, for each color i, c_i, for each color i and clique j, w_{ij},

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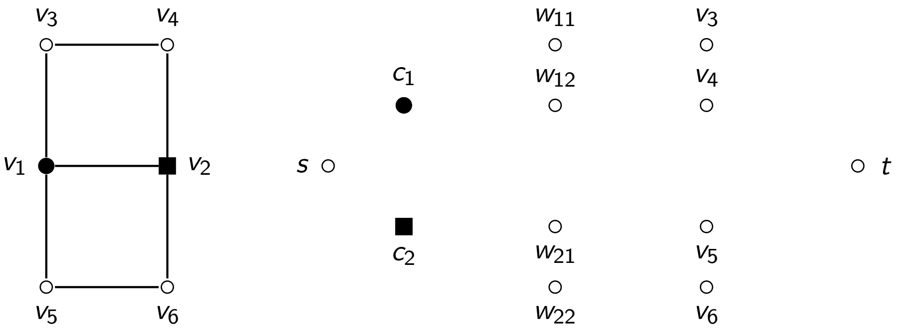

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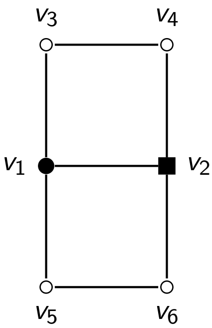

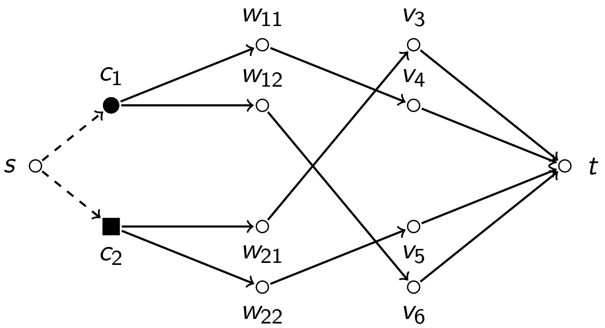

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Each flow unit gives the color of one vertex. Solid arcs have unit capacity.

Parameterized Complexity

The parameterized story so far...

Class	Parameterized Complexity
Bipartite	paraNP-hard parameterized by #colors
Cographs	W[1]-hard parameterized by #colors
Chordal	W[1]-hard parameterized by #colors
Block	?
Disjoint union of Split	?
$K_{1,4}$-free interval	?
Independent set $+kv$	FPT
Split $+kv$	W[1]-hard parameterized by k
Disjoint Union of Cliques $+kv$	?
Complete Multipartite $+kv$	?
Forest $+kv$	W[1]-hard parameterized by $k+$ #colors
Path $+kv$	?

In this talk

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Independent set $+kv$	FPT
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Cluster $+kv$	FPT param. by k
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Bin-packing is W[1]-hard parameterized by #bins.

Disjoint union of split graphs (complete *p*-partite)

Theorem

EQUITABLE COLORING *of disjoint union of split graphs is* W[1]-hard *when parameterized by number of colors and treewidth.*

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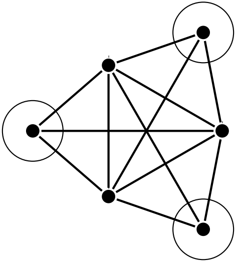

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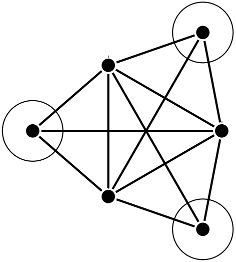
- $|V(G)| = \sum_{a_j \in A}(k-1)+(a_j+1) = k(n+B)$.



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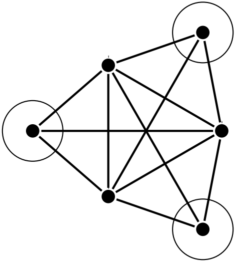
- $|V(G)| = \sum_{a_j \in A}(k-1)+(a_j+1) = k(n+B)$.
- Try equitably k-color it.



$K_{1,r}$-free interval graphs

Theorem

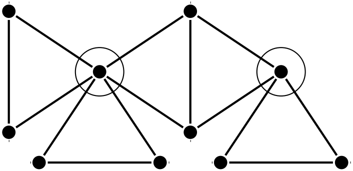
EQUITABLE COLORING *of* $K_{1,r}$*-free interval graphs is* W[1]-hard *when parameterized by number of colors, treewidth and maximum degree if* $r \geq 4$*, otherwise it is solvable in polynomial time (consequence of de Werra'85).*

$K_{1,r}$-free interval graphs

Theorem

Equitable Coloring of $K_{1,r}$-free interval graphs is W[1]-hard *when parameterized by number of colors, treewidth and maximum degree if $r \geq 4$, otherwise it is solvable in polynomial time (consequence of de Werra'85).*

- Each a_j becomes a sequence of a_j cliques of size $k - 1$. Add one universal vertex to each pair of consecutive cliques. Said vertex also has an extra clique of size $k - 1$ attached to it.



$k = 3 \quad a_j = 2$

$K_{1,r}$-free interval graphs

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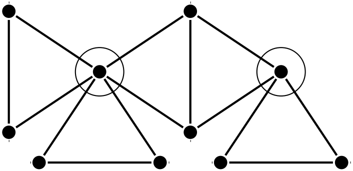

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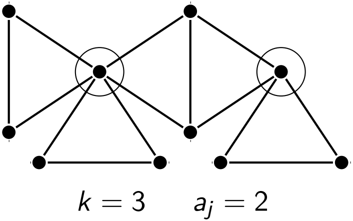
- Again, try to equitably k-color G.



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G is a cluster graph if each of its connected components is a clique (cluster).

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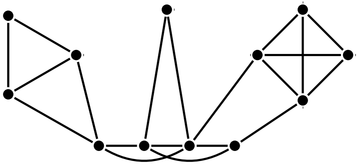

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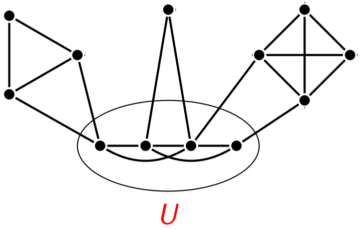

G is a cluster $+kv$ graph if there is a set $U \subset V(G)$ of size k such that $G - U$ is a cluster graph, with clusters $\{C_1, \ldots, C_\ell\}$.

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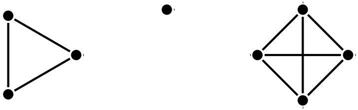

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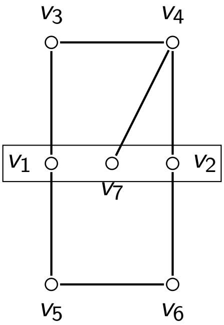

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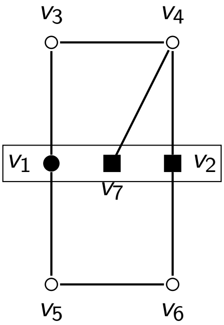

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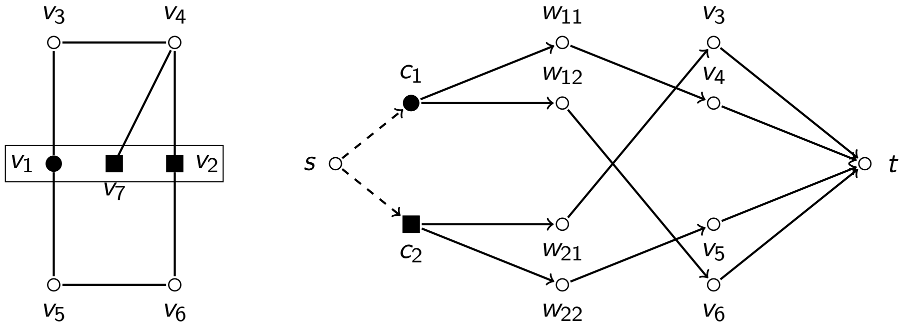
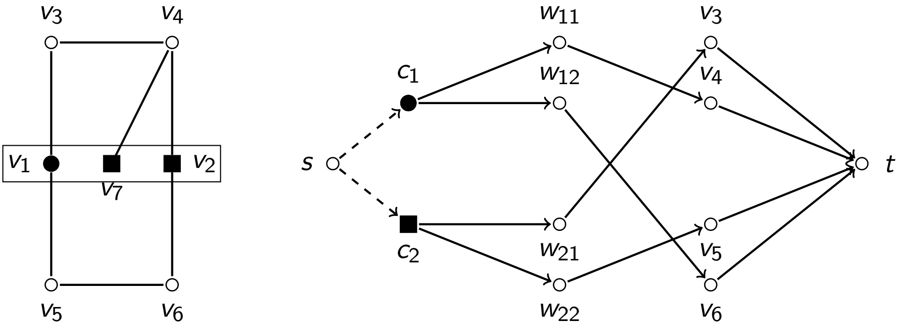


Algorithm

For each of the k^k colorings of U, construct the auxiliary graph.

Cluster $+kv$: max-flow again



Algorithm

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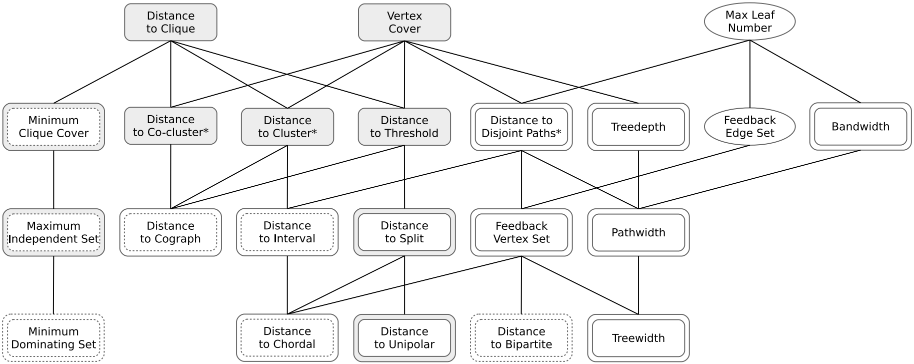
Parameterized landscape



Thank you!