The Combinatorial Complexity of Approximating Polytopes

Sunil Arya
Hong Kong University of Science and Technology

Guilherme D. da Fonseca
Université d'Auvergne and LIMOS

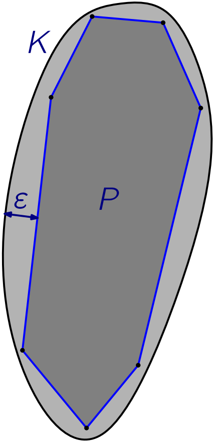
David M. Mount
University of Maryland, College Park

SoCG 2016

Polytope Approximation

Problem description:

- Input: convex body K in d-dimensional space and parameter ε
- Output: *succinct* polytope P which *ε-approximates* K
- Question: how succinct can P be?



- ε-approximate: Hausdorff distance $\varepsilon \cdot \text{diam}(K)$
- Succinct: Low *combinatorial complexity* (to be defined)
- Assume w.l.o.g. that $\text{diam}(K) = 1$
- Dimension d is a constant ($2^d/\varepsilon = O(1/\varepsilon)$)

Uniform vs. Nonuniform Bounds

Nonuniform bounds:

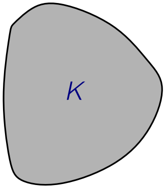
- Hold for $\varepsilon \leq \varepsilon_0$, where ε_0 depends on the input K
- Example: Gruber [Gru93] bounds the complexity n using the Gaussian curvature κ of the input

$$n = (1/\varepsilon)^{(d-1)/2} \int_{\partial K} \sqrt{\kappa(x)}\, dx$$

Uniform bounds: (our case)

- Hold for $\varepsilon \leq \varepsilon_0$, where ε_0 is a constant
- Example: Dudley [Dud74] and Bronshteyn and Ivanov [BI74] bound the maximum number of facets/vertices as a function of ε, d, and the diameter of the input

Bronshteyn and Ivanov's Approximation

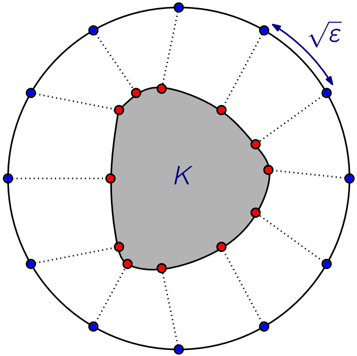


1. Surround K by a sphere of radius 2
2. Distribute points on the sphere with distance $\sim \sqrt{\varepsilon}$
3. Take the nearest neighbor on K for each point
4. Make P the convex hull of the points

Bronshteyn and Ivanov, 1974:

A convex body K of diameter 1 can be ε-approximated by a polytope P with $O(1/\varepsilon^{(d-1)/2})$ vertices.

Bronshteyn and Ivanov's Approximation

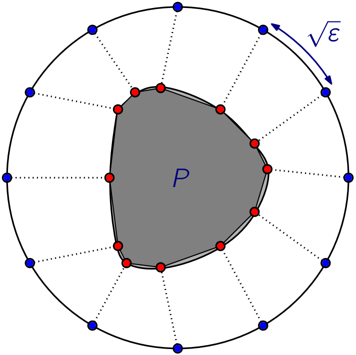


1. Surround K by a sphere of radius 2
2. Distribute points on the sphere with distance $\sim \sqrt{\varepsilon}$
3. Take the nearest neighbor on K for each point
4. Make P the convex hull of the points

Bronshteyn and Ivanov, 1974:

A convex body K of diameter 1 can be ε-approximated by a polytope P with $O(1/\varepsilon^{(d-1)/2})$ vertices.

Bronshteyn and Ivanov's Approximation

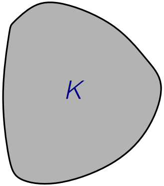


1. Surround K by a sphere of radius 2
2. Distribute points on the sphere with distance $\sim \sqrt{\varepsilon}$
3. Take the nearest neighbor on K for each point
4. Make P the convex hull of the points

Bronshteyn and Ivanov, 1974:

A convex body K of diameter 1 can be ε-approximated by a polytope P with $O(1/\varepsilon^{(d-1)/2})$ vertices.

Dudley's Approximation

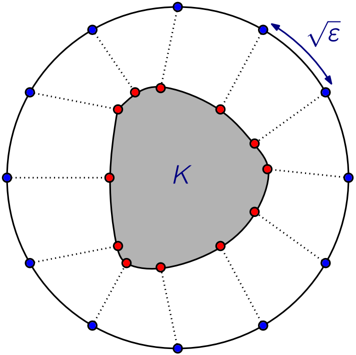


1. Surround K by a sphere of radius 2
2. Distribute points on the sphere with distance $\sim \sqrt{\varepsilon}$
3. Take the nearest neighbor on K for each point
4. Make P the intersection of supporting halfspaces at each point

Dudley, 1974:

A convex body K of diameter 1 can be ε-approximated by a polytope P with $O(1/\varepsilon^{(d-1)/2})$ facets.

Dudley's Approximation

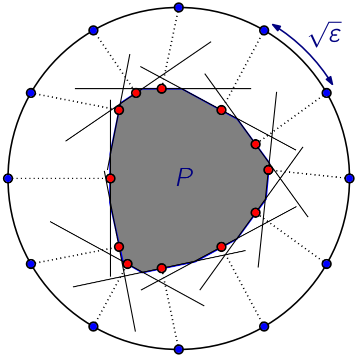


1. Surround K by a sphere of radius 2
2. Distribute points on the sphere with distance $\sim \sqrt{\varepsilon}$
3. Take the nearest neighbor on K for each point
4. Make P the intersection of supporting halfspaces at each point

Dudley, 1974:

A convex body K of diameter 1 can be ε-approximated by a polytope P with $O(1/\varepsilon^{(d-1)/2})$ facets.

Dudley's Approximation



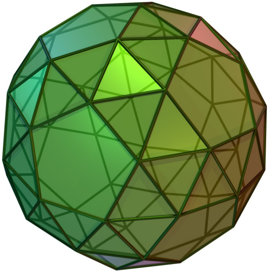
1. Surround K by a sphere of radius 2
2. Distribute points on the sphere with distance $\sim \sqrt{\varepsilon}$
3. Take the nearest neighbor on K for each point
4. Make P the intersection of supporting halfspaces at each point

Dudley, 1974:

A convex body K of diameter 1 can be ε-approximated by a polytope P with $O(1/\varepsilon^{(d-1)/2})$ facets.

Combinatorial Complexity

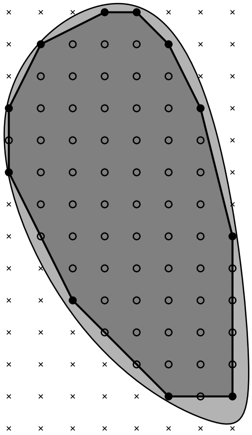
- Faces have different dimensions:
  0-face: *vertex*
  1-face: *edge*
  ⋮
  $(d-1)$-face: *facet*
- Combinatorial complexity:
  sum of number of k-faces $k = 0, \ldots, d-1$



Upper bound theorem

A polytope with n vertices (or facets) has combinatorial complexity $O(n^{\lfloor d/2 \rfloor})$.

Approximation of Low Combinatorial Complexity



Lattice polytope

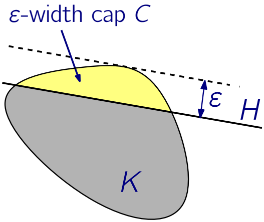
- Bronshteyn and Ivanov:
  optimal number of vertices $O(1/\varepsilon^{(d-1)/2})$
- Dudley: same optimal number of facets
- Upper bound theorem: their combinatorial complexity is $O(1/\varepsilon^{d^2/4})$
  (Maybe it's much better, but we don't know how to prove)

Combinatorial complexity

- Best known bound: Roughly $O(1/\varepsilon^{d-2})$ using lattice polytopes [And63]
- Our bound[a]: $\widetilde{O}(1/\varepsilon^{(d-1)/2})$

[a] $\widetilde{O}$ hides $\log^{(d-1)/2} \frac{1}{\varepsilon}$

Caps

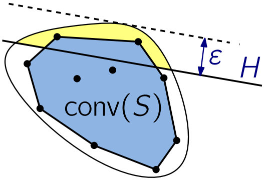


- Cap: intersection of K and a halfspace H
- Width: measured perpendicular to H
- A set S of points stabs all ε-width caps if for every ε-width cap C we have $C \cap S \neq \emptyset$

Approximation via Hitting Sets

If a point set $S \subset K$ stabs all ε-width caps, then $\text{conv}(S)$ is an ε-approximation to K.

Caps



- **Cap**: intersection of K and a halfspace H
- **Width**: measured perpendicular to H
- A set S of points **stabs** all ε-width caps if for every ε-width cap C we have $C \cap S \neq \emptyset$

Approximation via Hitting Sets

If a point set $S \subset K$ stabs all ε-width caps, then conv(S) is an ε-approximation to K.

Bounding Complexity via Witness-Collector

We identify two sets of regions:

- $\mathcal{W}$: witnesses
- $\mathcal{C}$: collectors, one per witness

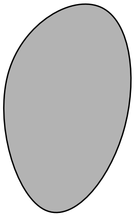
that satisfy:

(1) Each witness contributes one point to S

(2) Any halfspace H either:

- Deep: contains a witness, or
- Shallow: $H \cap K$ is contained within a collector

(3) Each collector contains $O(1)$ points of S



Witness-Collector Complexity Bound [Devillers et al. 2013]

Given a set of witnesses and collectors satisfying the above properties, the combinatorial complexity of the $\text{conv}(S)$ is $O(|\mathcal{C}|)$.

Bounding Complexity via Witness-Collector

We identify two sets of regions:

- $\mathcal{W}$: witnesses
- $\mathcal{C}$: collectors, one per witness

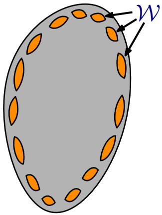
that satisfy:

(1) Each witness contributes one point to S

(2) Any halfspace H either:

  - Deep: contains a witness, or
  - Shallow: $H \cap K$ is contained within a collector

(3) Each collector contains $O(1)$ points of S



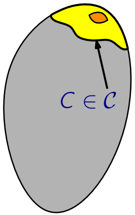
Witness-Collector Complexity Bound [Devillers et al. 2013]

Given a set of witnesses and collectors satisfying the above properties, the combinatorial complexity of the conv(S) is $O(|\mathcal{C}|)$.

Bounding Complexity via Witness-Collector

We identify two sets of regions:

- $\mathcal{W}$: witnesses
- $\mathcal{C}$: collectors, one per witness



that satisfy:

(1) Each witness contributes one point to S

(2) Any halfspace H either:

- Deep: contains a witness, or
- Shallow: $H \cap K$ is contained within a collector

(3) Each collector contains $O(1)$ points of S

Witness-Collector Complexity Bound [Devillers et al. 2013]

Given a set of witnesses and collectors satisfying the above properties, the combinatorial complexity of the conv(S) is $O(|\mathcal{C}|)$.

Bounding Complexity via Witness-Collector

We identify two sets of regions:

- $\mathcal{W}$: witnesses
- $\mathcal{C}$: collectors, one per witness

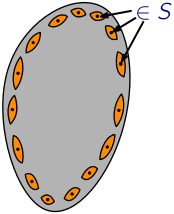
that satisfy:

(1) Each witness contributes one point to S

(2) Any halfspace H either:

- Deep: contains a witness, or
- Shallow: $H \cap K$ is contained within a collector

(3) Each collector contains $O(1)$ points of S



Witness-Collector Complexity Bound [Devillers et al. 2013]

Given a set of witnesses and collectors satisfying the above properties, the combinatorial complexity of the $\text{conv}(S)$ is $O(|\mathcal{C}|)$.

Bounding Complexity via Witness-Collector

We identify two sets of regions:

- $\mathcal{W}$: witnesses
- $\mathcal{C}$: collectors, one per witness

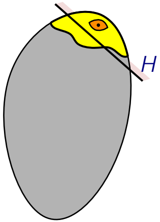
that satisfy:

(1) Each witness contributes one point to S

(2) Any halfspace H either:

  - Deep: contains a witness, or
  - Shallow: $H \cap K$ is contained within a collector

(3) Each collector contains $O(1)$ points of S



Witness-Collector Complexity Bound [Devillers et al. 2013]

Given a set of witnesses and collectors satisfying the above properties, the combinatorial complexity of the conv(S) is $O(|\mathcal{C}|)$.

Bounding Complexity via Witness-Collector

We identify two sets of regions:

- $\mathcal{W}$: witnesses
- $\mathcal{C}$: collectors, one per witness

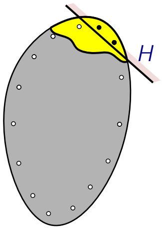
that satisfy:

(1) Each witness contributes one point to S

(2) Any halfspace H either:

  - Deep: contains a witness, or
  - Shallow: $H \cap K$ is contained within a collector

(3) Each collector contains $O(1)$ points of S



Witness-Collector Complexity Bound [Devillers et al. 2013]

Given a set of witnesses and collectors satisfying the above properties, the combinatorial complexity of the conv(S) is $O(|\mathcal{C}|)$.

Bounding Complexity via Witness-Collector

We identify two sets of regions:

- $\mathcal{W}$: witnesses
- $\mathcal{C}$: collectors, one per witness

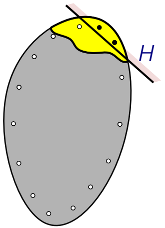
that satisfy:

(1) Each witness contributes one point to S

(2) Any halfspace H either:

  - Deep: contains a witness, or
  - Shallow: $H \cap K$ is contained within a collector

(3) Each collector contains $O(1)$ points of S



Witness-Collector Complexity Bound [Devillers et al. 2013]

Given a set of witnesses and collectors satisfying the above properties, the combinatorial complexity of the conv(S) is $O(|\mathcal{C}|)$.

Bounding Complexity via Witness-Collector

We identify two sets of regions:

- $\mathcal{W}$: witnesses
- $\mathcal{C}$: collectors, one per witness

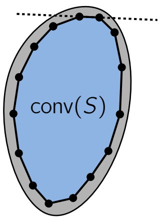
that satisfy:

(1) Each witness contributes one point to S

(2) Any halfspace H either:

  - Deep: contains a witness, or
  - Shallow: $H \cap K$ is contained within a collector

(3) Each collector contains $O(1)$ points of S



Witness-Collector Complexity Bound [Devillers et al. 2013]

Given a set of witnesses and collectors satisfying the above properties, the combinatorial complexity of the conv(S) is $O(|\mathcal{C}|)$.

Bounding Complexity via Witness-Collector

We identify two sets of regions:

- $\mathcal{W}$: witnesses
- $\mathcal{C}$: collectors, one per witness

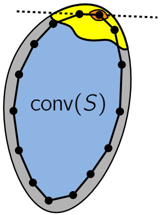
that satisfy:

(1) Each witness contributes one point to S

(2) Any halfspace H either:

  - Deep: contains a witness, or
  - Shallow: $H \cap K$ is contained within a collector

(3) Each collector contains $O(1)$ points of S



Witness-Collector Complexity Bound [Devillers et al. 2013]

Given a set of witnesses and collectors satisfying the above properties, the combinatorial complexity of the $\text{conv}(S)$ is $O(|\mathcal{C}|)$.

Bounding Complexity via Witness-Collector

We identify two sets of regions:

- $\mathcal{W}$: witnesses
- $\mathcal{C}$: collectors, one per witness

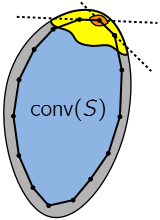
that satisfy:

(1) Each witness contributes one point to S

(2) Any halfspace H either:

- Deep: contains a witness, or
- Shallow: $H \cap K$ is contained within a collector

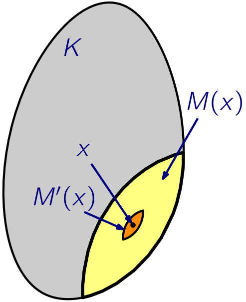
(3) Each collector contains $O(1)$ points of S



Witness-Collector Complexity Bound [Devillers et al. 2013]

Given a set of witnesses and collectors satisfying the above properties, the combinatorial complexity of the conv(S) is $O(|\mathcal{C}|)$.

Macbeath Regions



Macbeath Regions [Macbeath 52]

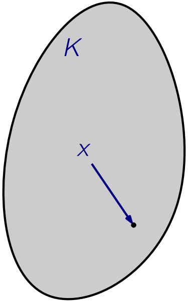
Given a convex body K, $x \in K$, and $\lambda > 0$:

- $M^\lambda(x) = x + \lambda((K - x) \cap (x - K))$
- $M(x) = M^1(x)$: intersection of K and K reflected around x
- $M'(x) = M^{1/5}(x)$

Properties:

- $M(x)$ resembles the minimum volume cap containing x
- $M'(x) \cap M'(y) \neq \emptyset \Rightarrow M'(x) \subseteq M(y)$

Macbeath Regions



Macbeath Regions [Macbeath 52]

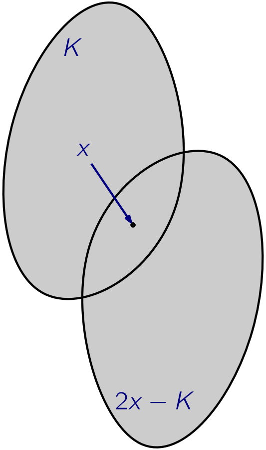
Given a convex body K, $x \in K$, and $\lambda > 0$:

- $M^{\lambda}(x) = x + \lambda((K - x) \cap (x - K))$
- $M(x) = M^{1}(x)$: intersection of K and K reflected around x
- $M'(x) = M^{1/5}(x)$

Properties:

- $M(x)$ resembles the minimum volume cap containing x
- $M'(x) \cap M'(y) \neq \emptyset \;\Rightarrow\; M'(x) \subseteq M(y)$

Macbeath Regions



Macbeath Regions [Macbeath 52]

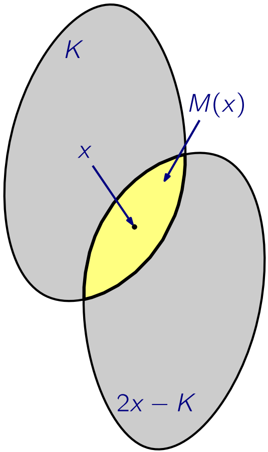
Given a convex body K, $x \in K$, and $\lambda > 0$:

- $M^{\lambda}(x) = x + \lambda((K - x) \cap (x - K))$
- $M(x) = M^{1}(x)$: intersection of K and K reflected around x
- $M'(x) = M^{1/5}(x)$

Properties:

- $M(x)$ resembles the minimum volume cap containing x
- $M'(x) \cap M'(y) \neq \emptyset \;\Rightarrow\; M'(x) \subseteq M(y)$

Macbeath Regions



Macbeath Regions [Macbeath 52]

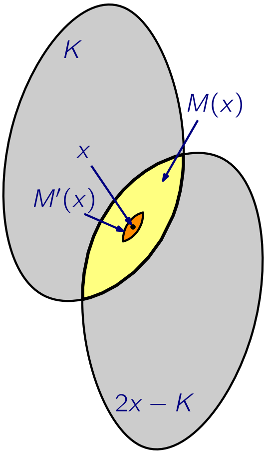
Given a convex body K, $x \in K$, and $\lambda > 0$:

- $M^{\lambda}(x) = x + \lambda((K - x) \cap (x - K))$
- $M(x) = M^1(x)$: intersection of K and K reflected around x
- $M'(x) = M^{1/5}(x)$

Properties:

- $M(x)$ resembles the minimum volume cap containing x
- $M'(x) \cap M'(y) \neq \emptyset \Rightarrow M'(x) \subseteq M(y)$

Macbeath Regions



Macbeath Regions [Macbeath 52]

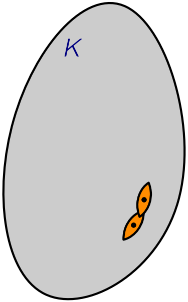
Given a convex body K, $x \in K$, and $\lambda > 0$:

- $M^\lambda(x) = x + \lambda((K - x) \cap (x - K))$
- $M(x) = M^1(x)$: intersection of K and K reflected around x
- $M'(x) = M^{1/5}(x)$

Properties:

- $M(x)$ resembles the minimum volume cap containing x
- $M'(x) \cap M'(y) \neq \emptyset \Rightarrow M'(x) \subseteq M(y)$

Macbeath Regions



Macbeath Regions [Macbeath 52]

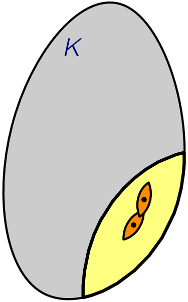
Given a convex body K, $x \in K$, and $\lambda > 0$:

- $M^{\lambda}(x) = x + \lambda((K - x) \cap (x - K))$
- $M(x) = M^{1}(x)$: intersection of K and K reflected around x
- $M'(x) = M^{1/5}(x)$

Properties:

- $M(x)$ resembles the minimum volume cap containing x
- $M'(x) \cap M'(y) \neq \emptyset \;\Rightarrow\; M'(x) \subseteq M(y)$

Macbeath Regions



Macbeath Regions [Macbeath 52]

Given a convex body K, $x \in K$, and $\lambda > 0$:

- $M^\lambda(x) = x + \lambda((K - x) \cap (x - K))$
- $M(x) = M^1(x)$: intersection of K and K reflected around x
- $M'(x) = M^{1/5}(x)$

Properties:

- $M(x)$ resembles the minimum volume cap containing x
- $M'(x) \cap M'(y) \neq \emptyset \;\Rightarrow\; M'(x) \subseteq M(y)$

Economical Cap Covering

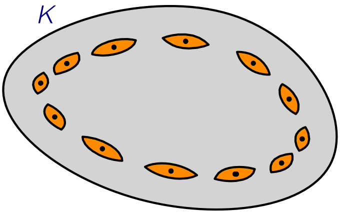
Economical Cap Covering

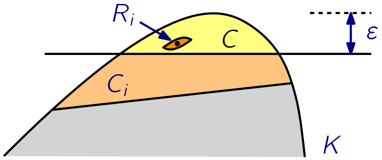
Given:

- K: convex body
- ε: small positive parameter

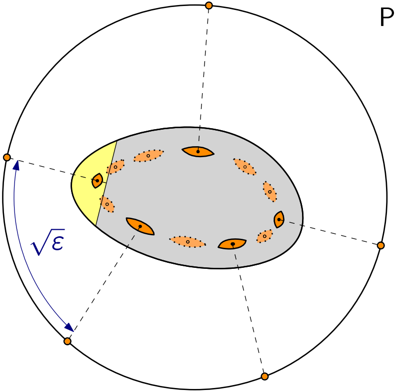
There exists [ELR70, Bar07]:

- Macbeath regions $R_1, \ldots, R_k$
- Caps $C_1, \ldots, C_k$ of width $\Theta(\varepsilon)$
- $k = O(1/\varepsilon^{(d-1)/2})$ (new)
- For every cap C, there is i:
  - $R_i \subseteq C$ if width $\geq \varepsilon$
  - $C \subseteq C_i$ if width $\leq \varepsilon$





Cardinality Bound



Proof strategy:

- Prune Macbeath regions that are too close to each other (by increasing volume)
- A constant fraction of the regions are pruned
- Project the centers of the regions onto the Dudley ball (perpendicularly to the corresponding cap)
- Show that the pairwise distance in the Dudley ball is at least $\sqrt{\varepsilon}$

Approximating with ECC

Economical Cap Covering

Given:

- K: convex body
- ε: small positive parameter

There exists [ELR70, Bar07]:

- Macbeath regions $R_1, \ldots, R_k$
- Caps $C_1, \ldots, C_k$ of width $\Theta(\varepsilon)$
- $k = O(1/\varepsilon^{(d-1)/2})$ (new)
- For every cap C, there is i:
  - $R_i \subseteq C$ if width $\geq \varepsilon$
  - $C \subseteq C_i$ if width $\leq \varepsilon$

Procedure:

- Create an ECC
- Place a point inside each Macbeath region
- Compute the convex hull

Approximation:

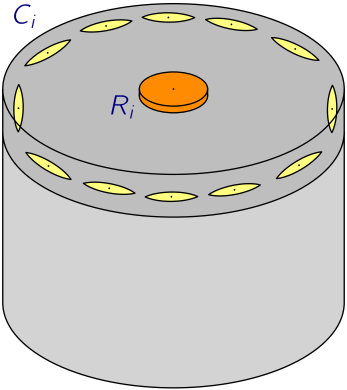
- Every ε-width cap is stabbed

Complexity:

- $O(1/\varepsilon^{(d-1)/2})$ *vertices*
- Combinatorial complexity: $O(1/\varepsilon^{d^2/4})$ by the upper bound theorem

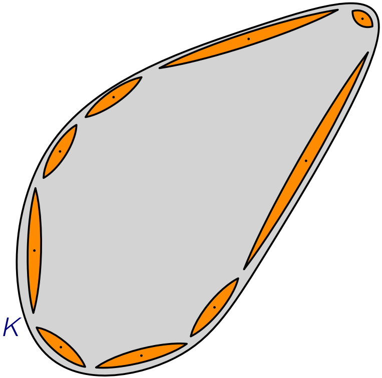
Better Bound on the Combinatorial Complexity

What if we use the Macbeath regions as witnesses and the caps as collectors?



- Does not work for the traditional construction (for fixed width)
- We need one additional property: For all i, C_i intersects a constant number of bodies R_j

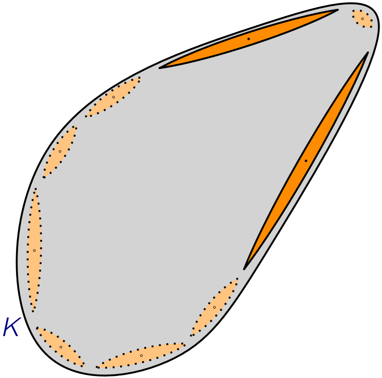
Stratified Construction



- Organize the Macbeath regions in $O(\log \frac{1}{\varepsilon})$ layers by *volume*
- Larger volume Macbeath regions in outer layers
- Approximation error increases from ε to $O(\varepsilon \log \frac{1}{\varepsilon}))$
- Scale ε accordingly: $\hat{\varepsilon} = \frac{\varepsilon}{\log \frac{1}{\varepsilon}}$
- Number of regions grows to

$$O\left(\left(\frac{1}{\hat{\varepsilon}}\right)^{\frac{d-1}{2}}\right) = O\left(\left(\frac{\log \frac{1}{\varepsilon}}{\varepsilon}\right)^{\frac{d-1}{2}}\right)$$

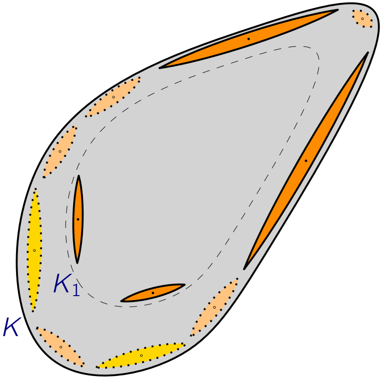
Stratified Construction



- Organize the Macbeath regions in $O(\log \frac{1}{\varepsilon})$ layers by *volume*
- Larger volume Macbeath regions in outer layers
- Approximation error increases from ε to $O(\varepsilon \log \frac{1}{\varepsilon}))$
- Scale ε accordingly: $\hat{\varepsilon} = \frac{\varepsilon}{\log \frac{1}{\varepsilon}}$
- Number of regions grows to

$$O\left(\left(\frac{1}{\hat{\varepsilon}}\right)^{\frac{d-1}{2}}\right) = O\left(\left(\frac{\log \frac{1}{\varepsilon}}{\varepsilon}\right)^{\frac{d-1}{2}}\right)$$

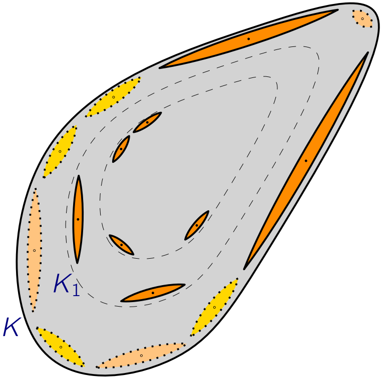
Stratified Construction



- Organize the Macbeath regions in $O(\log \frac{1}{\varepsilon})$ layers by *volume*
- Larger volume Macbeath regions in outer layers
- Approximation error increases from ε to $O(\varepsilon \log \frac{1}{\varepsilon}))$
- Scale ε accordingly: $\hat{\varepsilon} = \frac{\varepsilon}{\log \frac{1}{\varepsilon}}$
- Number of regions grows to

$$O\left(\left(\frac{1}{\hat{\varepsilon}}\right)^{\frac{d-1}{2}}\right) = O\left(\left(\frac{\log \frac{1}{\varepsilon}}{\varepsilon}\right)^{\frac{d-1}{2}}\right)$$

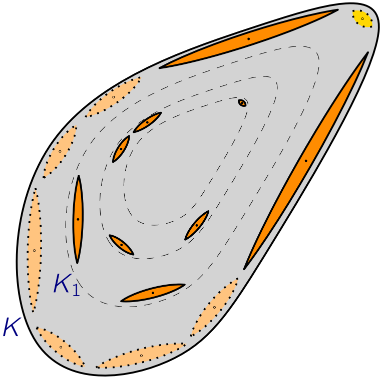
Stratified Construction



- Organize the Macbeath regions in $O(\log \frac{1}{\varepsilon})$ layers by *volume*
- Larger volume Macbeath regions in outer layers
- Approximation error increases from ε to $O(\varepsilon \log \frac{1}{\varepsilon}))$
- Scale ε accordingly: $\hat{\varepsilon} = \frac{\varepsilon}{\log \frac{1}{\varepsilon}}$
- Number of regions grows to

$$O\left(\left(\frac{1}{\hat{\varepsilon}}\right)^{\frac{d-1}{2}}\right) = O\left(\left(\frac{\log \frac{1}{\varepsilon}}{\varepsilon}\right)^{\frac{d-1}{2}}\right)$$

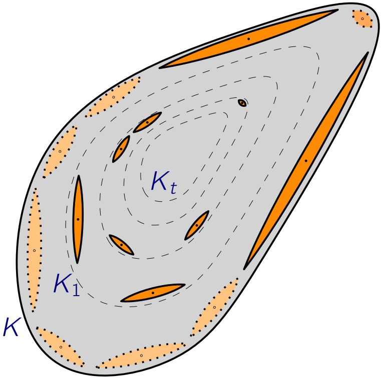
Stratified Construction



- Organize the Macbeath regions in $O(\log \frac{1}{\varepsilon})$ layers by *volume*
- Larger volume Macbeath regions in outer layers
- Approximation error increases from ε to $O(\varepsilon \log \frac{1}{\varepsilon}))$
- Scale ε accordingly: $\hat{\varepsilon} = \frac{\varepsilon}{\log \frac{1}{\varepsilon}}$
- Number of regions grows to

$$O\left(\left(\frac{1}{\hat{\varepsilon}}\right)^{\frac{d-1}{2}}\right) = O\left(\left(\frac{\log \frac{1}{\varepsilon}}{\varepsilon}\right)^{\frac{d-1}{2}}\right)$$

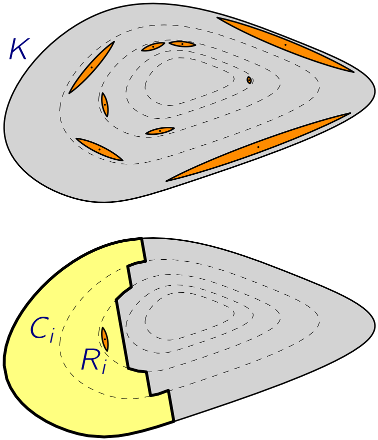
Stratified Construction



- Organize the Macbeath regions in $O(\log \frac{1}{\varepsilon})$ layers by *volume*
- Larger volume Macbeath regions in outer layers
- Approximation error increases from ε to $O(\varepsilon \log \frac{1}{\varepsilon}))$
- Scale ε accordingly: $\hat{\varepsilon} = \frac{\varepsilon}{\log \frac{1}{\varepsilon}}$
- Number of regions grows to

$$O\left(\left(\frac{1}{\hat{\varepsilon}}\right)^{\frac{d-1}{2}}\right) = O\left(\left(\frac{\log \frac{1}{\varepsilon}}{\varepsilon}\right)^{\frac{d-1}{2}}\right)$$

Stratified Economical Cap Covering



Stratified ECC

Given:

- K: convex body
- ε: small positive parameter

There exists:

- Convex bodies $R_1, \ldots, R_k$
- Regions $C_1, \ldots, C_k$
- $k = \widetilde{O}(1/\varepsilon^{(d-1)/2})$
- Every ε-width cap C contains an R_i
- For every cap C, either $R_i \subseteq C$ or $C \subseteq C_i$ for some i
- For all i, C_i intersects a constant number of bodies R_j

Comparing Economical Cap Coverings

Economical Cap Covering

Given: K, ε

There exists:

- Convex bodies $R_1, \ldots, R_k$
- Caps $C_1, \ldots, C_k$ of width $\Theta(\varepsilon)$
- $k = O(1/\varepsilon^{(d-1)/2})$
- For every cap C, there is i:
  - $R_i \subseteq C$ if width $\geq \varepsilon$
  - $C \subseteq C_i$ if width $\leq \varepsilon$

Stratified ECC

Given: K, ε

There exists:

- Convex bodies $R_1, \ldots, R_k$
- Non-convex regions $C_1, \ldots, C_k$
- $k = \widetilde{O}(1/\varepsilon^{(d-1)/2})$
- For every cap C, there is i:
  - $R_i \subseteq C$ if width $\geq \varepsilon$
  - $C \subseteq C_i$ if no $R_j \subseteq C$
- For all i, C_i intersects a constant number of bodies R_j

Succinct Approximation

Procedure:

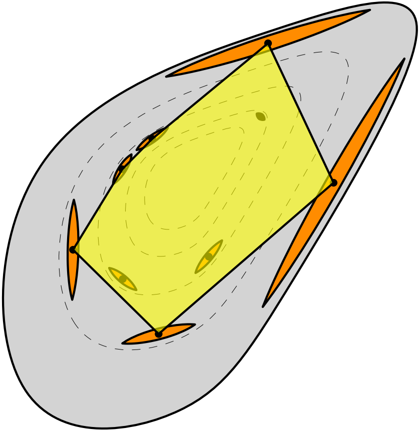
- Create a stratified ECC
- Place a point inside each R_i
- Compute the convex hull

Approximation:

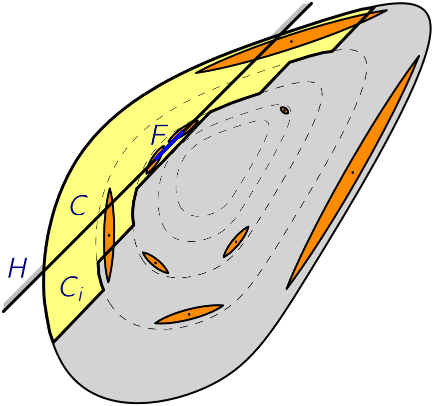
- Every ε-width cap is stabbed

Complexity:

- $\widetilde{O}(1/\varepsilon^{(d-1)/2})$ vertices
- Combinatorial complexity: $\widetilde{O}(1/\varepsilon^{(d-1)/2})$ Why?



Counting the Faces



- R_i are witnesses and C_i are collectors
- Charge each face F to a C_i and show each C_i receives $O(1)$ charges
- H: supporting halfspace of F
- $C = K \cap H$ contains no R_i
- Some C_i contains C
- Charge F to C_i
- C_i intersects $O(1)$ bodies R_i
  $\rightarrow$ C_i contains $O(1)$ points
  $\rightarrow$ $O(1)$ faces charge C_i

Conclusion and Open Problems

Our result:

First near-optimal bound on the combinatorial complexity of an ε-approximating polytope in $\mathbb{R}^d$:

$$O\left(\left(\frac{\log\frac{1}{\varepsilon}}{\varepsilon}\right)^{\frac{d-1}{2}}\right) = \widetilde{O}\left(\left(\frac{1}{\varepsilon}\right)^{\frac{d-1}{2}}\right)$$

Open problems:

- How long does it take to actually build the approximation?
- Can we analyze Dudley's construction?
- Can we get rid of the $\log$ factors?
- Can we compete against the optimal combinatorial complexity for the given K?

Bibliography

[And63] G. E. Andrews. A lower bound for the volumes of strictly convex bodies with many boundary points. *Trans. Amer. Math. Soc.*, 106:270–279, 1963.

[Bar07] I. Bárány. Random polytopes, convex bodies, and approximation. In W. Weil, editor, *Stochastic Geometry*, volume 1892 of *Lecture Notes in Mathematics*, pages 77–118, 2007.

[BI74] E. M. Bronshteyn and L. D. Ivanov. The approximation of convex sets by polyhedra. *Siberian Math. J.*, 16:852–853, 1976. (original in Russian: 1974.)

[DGG13] O. Devillers, M. Glisse, and X. Goaoc. Complexity analysis of random geometric structures made simpler. In *Proc. 29th Annu. Sympos. Comput. Geom.*, pages 167–176, 2013.

[Dud74] R. M. Dudley. Metric entropy of some classes of sets with differentiable boundaries. *Approx. Theory*, 10(3):227–236, 1974.

[ELR70] G. Ewald, D. G. Larman, and C. A. Rogers. The directions of the line segments and of the r-dimensional balls on the boundary of a convex body in Euclidean space. *Mathematika*, 17:1–20, 1970.

[Gru93] P. M. Gruber. Asymptotic estimates for best and stepwise approximation of convex bodies. *I. Forum Math.*, 5:521–537, 1993.

[Mac52] A. M. Macbeath. A theorem on non-homogeneous lattices. *Annals of Mathematics*, 54:431–438, 1952.

Sculpture by Antony Gormley

Thank you!