Combinatorial Games in Graphs: Timber Game and Coloring Game

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IME-UFF

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Summary

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Goal

- Many researchers have been studying winning strategies in 2-player combinatorial games.
- We study the Timber Game, Coloring Game and their structural properties in a caterpillar.
- Moreover, we study the Nordhaus-Gaddum type inequality to the parameters of these games.

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- Moreover, we study the Nordhaus-Gaddum type inequality to the parameters of these games.

Why games?

Figure: Salon International de la Culture et des jeux mathématiques, Paris, 2015.

Figure: Festival da Matemática, Rio de Janeiro, 2017.

Timber Game

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- In 1984, a video game called *timber* was released.
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- Timber is played on a digraph $D = (V, \vec{E})$, with a domino on each arc.
- If one domino is toppled, it topples the dominoes in the direction it was toppled and creates a chain reaction.
- The orientation of the arc represents the available movement of the domino piece.

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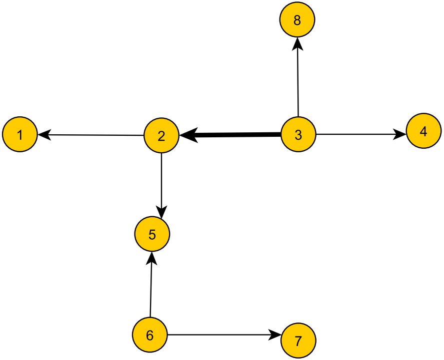
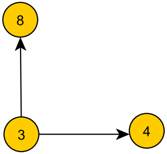
How to play?





What remains after toppling (3,2)

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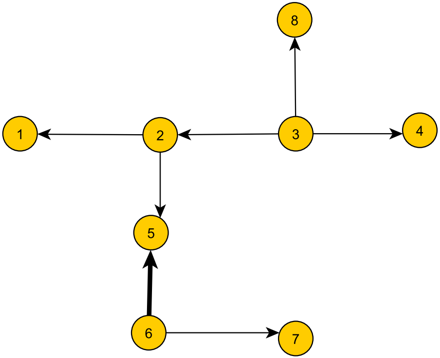

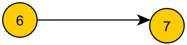

What remains after toppling (6,5)

Who wins?

- The player who topples all the last dominoes wins.
- In the last example, player 1 wins if he topples arc $(3, 2)$:

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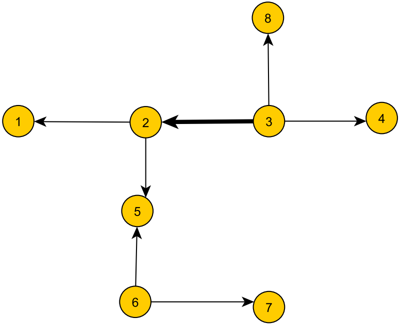

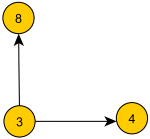

What remains after toppling (3,2)

P-position

- A P-position is a configuration D in which the second player wins, independently of how the first player plays.
- The last example is not a P-position, because there is a winning strategy for the first player.
- An oriented cycle is not a P-position.
- The study of Timber Game is only interesting in trees.

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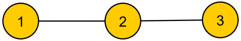
Example

Path G with 3 vertices:



Configurations of G:

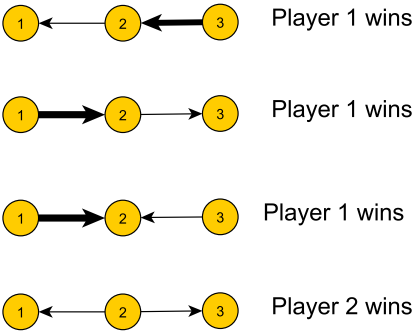


Figure: There is just 1 P-position.

Known results for paths (Nowakowski et al., 2013)

- Considering isomorphisms, we have:

edges (m)	1	2	3	4	5	6	7	8	9	10
P-positions	0	1	0	2	0	5	0	14	0	42

- Then, the number of P-positions of a path with m edges is given by:
  - 0 ; if m is odd;
  - $\frac{(m)!}{(\frac{m+2}{2})!(\frac{m}{2})!}$; if m is even.

- *Sequence of Catalan.*

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Known results for trees (Nowakowski et al., 2013): Lemma 1 - a decision lemma

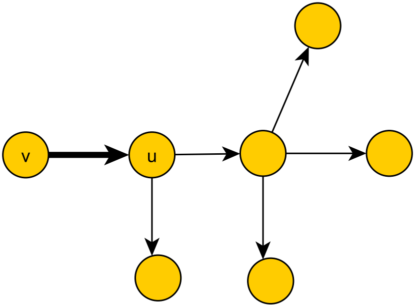

Figure: The first player wins toppling the piece (v, u)

Known results for trees (Nowakowski et al., 2013): Lemmas 2 and 3 - reduction lemmas

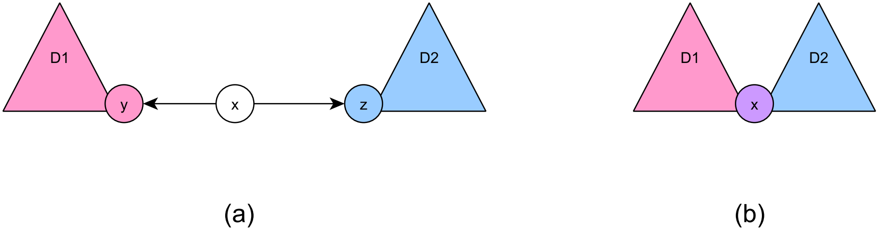

Figure: The digraph in (a) is a *P*-position iff the digraph in (b) is a *P*-position.

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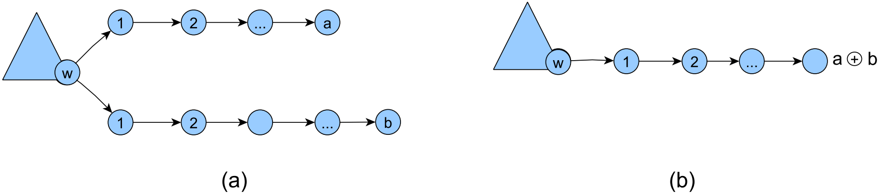

Figure: The digraph in (a) is a P-position iff the digraph in (b) is a P-position.

The importance of theses Lemmas

- These three lemmas compose the steps of a polynomial algorithm to decide if an oriented tree is or is not a P-position, presented in Nowakowski et al. (2013).

Theorem [DAM 2017]

Theorem

A tree has a P-position if, and only if, it has an even number of edges.

- ($\Rightarrow$) If a tree has a P-position, then the configuration that is a P-position can be reduced to a single vertex (0 arcs), by Lemmas 2.7 and 2.8. But Lemmas 2.7 and 2.8 maintain the parity of the number of edges.
- ($\Leftarrow$) We have an algorithm that assures us that there is always at least 1 P-position.

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Question

- What a happy surprise! As with paths, in a tree, if the number of edges is odd, then there is no P-position.
- Is there a unique formula to provide us the number of P-positions of a tree?

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What will we do?

In order to contribute to the open problem of determining the number of P-positions of a tree, we study the case when G is a caterpillar.

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- A *caterpillar* $cat(k_1, k_2, \ldots, k_s)$ is a tree which is obtained from a central path $v_1, v_2, v_3, ..., v_s$ (called spine), and by joining v_i to k_i new vertices, $i = 1, \ldots, s$. Thus, the number of vertices is $n = s + k_1 + k_2 + \ldots + k_s$.

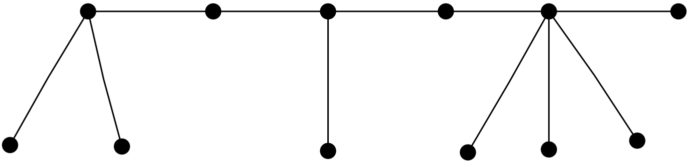

Figure: $cat(2, 0, 1, 0, 3, 0)$.

Theorem [DAM 2017]

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Let H be a caterpillar $cat(k_1, \ldots, k_s)$, for $k_i \in \mathbb{Z}, i = 1, \ldots, s$. The number of P-positions of H is equal to the number of P-positions of a caterpillar $cat(l_1, \ldots, l_s)$, such that if k_i is even, then $l_i = 0$, and if k_i is odd, then $l_i = 1$, for $i = 1, \ldots, s$.

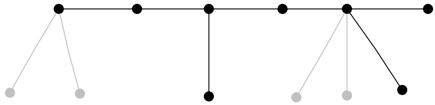

Figure: $cat(2, 0, 1, 0, 3, 0)$ is equivalent to $cat(0, 0, 1, 0, 1, 0)$.

Goal of the study in caterpillars

- We want to determine the number of P-positions of any caterpillar.

- We know that if the number of edges in a tree is odd, then the tree does not have P-positions.

- So let's investigate only caterpillars with an even number of edges.

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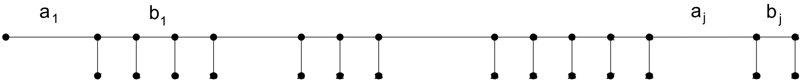

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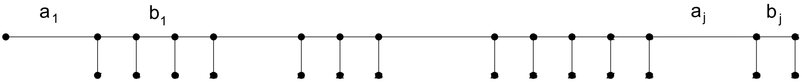

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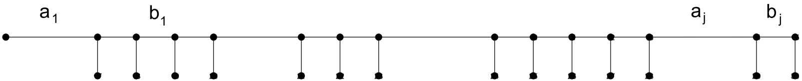

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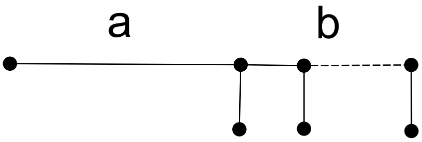
Family 1: caterpillar to solve the general case (?)



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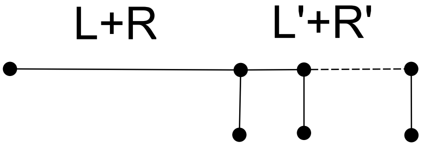

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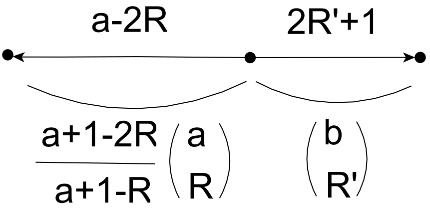

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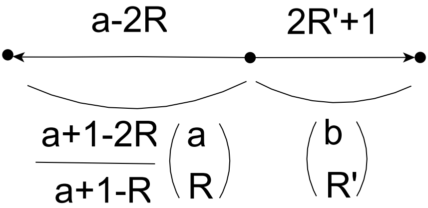

Theorem

If H is $cat(k_1, \ldots, k_a, \ldots, k_{a+b+1})$, such that $k_1, \ldots, k_a$ are even, $k_{a+1}, \ldots, k_{a+b+1}$ are odd, a is odd, and $b \geq 1$, then H has $\sum_{R'=0}^{b} \frac{4R'+4}{a+2R'+3} \binom{a}{\frac{a-2R'-1}{2}} \binom{b}{R'}$ P-positions.

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General case of caterpillar: a lower bound

How to use the previous caterpillar to solve the general case?

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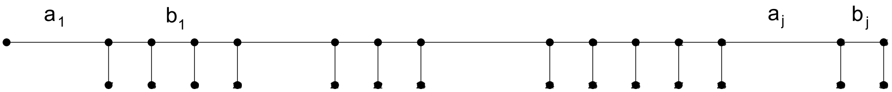

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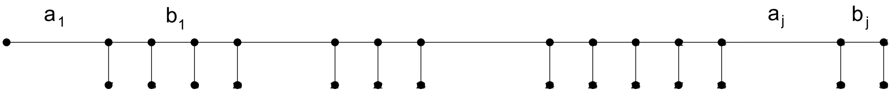

Theorem

A caterpillar cat $\langle a_1, b_1; a_2, b_2; \ldots; a_j, b_j\rangle$ *as in the Figure above has at least*

$\prod_{i=1}^{j} \sum_{R'_i=0}^{b_i} \frac{4R'_i+4}{a_i+2R'+3} \binom{a_i}{\frac{a_i-2R'_i-1}{2}} \binom{b_i}{R'_i}$ *P-positions, where* R'_i *is the number of edges oriented to the right among the* b_i *edges in the spine.*

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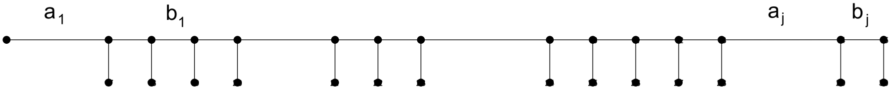

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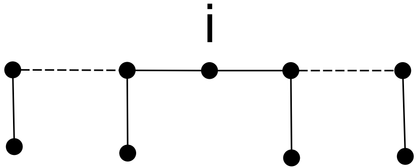
P-positions, where R'_i *is the number of edges oriented to the right among the* b_i *edges in the spine.*

This lower bound is tight.

Family 2: caterpillar without a leg



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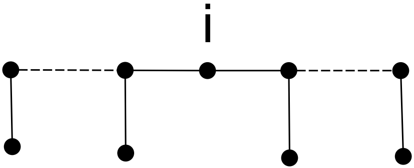

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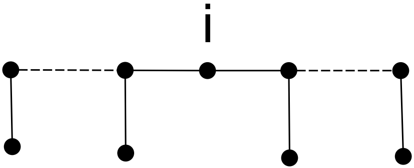
Let H be a caterpillar $cat(k_1, \ldots, k_s)$, *such that* k_i *is even and* $k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_s$ *are odd, for* $i = 1, \ldots, s$. *The number of P-positions of H is* $\binom{s-1}{i-1}$.



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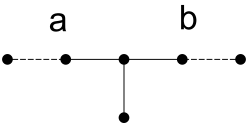
Proof by induction in s.

Family 3: caterpillar with just one leg



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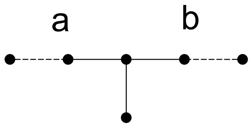

Theorem

If H is $cat(k_1, \ldots, k_{a+1}, \ldots, k_{a+b+1})$, such that only k_{a+1} is odd, $a, b \geq 1$ and $a+b+1$ is even, then H has

$$\sum_{R'=\lceil \frac{b}{2} \rceil}^{b} \frac{-2b+4R'+2+2(-1)^b}{a-b+2R'+2+(-1)^b} \binom{a}{\frac{a+b-2R'-(-1)^b}{2}} \frac{-b+2R'+1}{R'+1} \binom{b}{R'}$$

P-positions.

Comparing the number of P-positions

Caterpillar	Number of P-positions
P_s; s odd	$\cong \frac{2^s}{s^{2/3}}$
Family 1	$\cong s^{\frac{i-1}{2}}$
Family 2	$\cong \frac{s^{i-1}}{(i-1)!}$
Family 3	$\geq \frac{2^s}{(i(s-1))^{2/3}}$

Table: Comparing the number of P-positions

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Comparison between the number of P-positions of a caterpillar of Family 2 and a path

The graph below shows in the highlighted region for which values of s and i the caterpillar of Family 2 has more P-positions than the path P_{s+1}, when s is even (a), and more P-positions than the path P_{s+2}, when s is odd (b).

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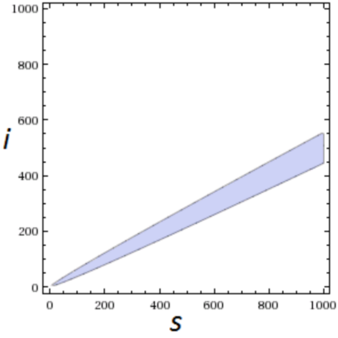
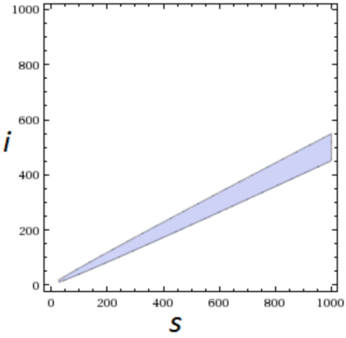


(a)



(b)

Conclusion: Timber Game

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It is a reduction problem.

We are able to determine the number of P-positions for infinite families of caterpillars.

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Presentations, publications and submissions

Furtado, A., Dantas, S., Figueiredo, C., Gravier, S., *Timber Game with Caterpillars.* Matemática Contemporânea 44 (2015), 1-9.

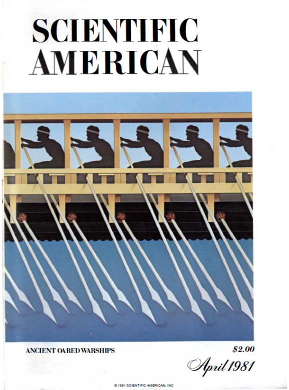
Furtado, A., Dantas, S., Figueiredo, C., Gravier, S., *Timber Game with Caterpillars.* In proceedings of the 13th Cologne-Twente Workshop on Graphs & Combinatorial Optimization, Istambul (2015).

Furtado, A., Dantas, S., Figueiredo, C., Gravier, S., *Timber Game as a counting problem.* Discrete Applied Mathematics special issue of GO X (2017).

Coloring Game

What is Coloring Game?

- The *coloring game* is a two player non-cooperative game conceived by Steven Brams.
- Firstly published in 1981 by Martin Gardner.
- Reinvented in 1991 by Bodlaender, who studied the game in the context of graphs.



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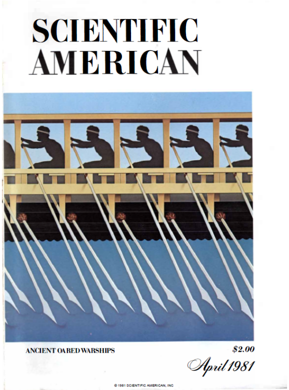

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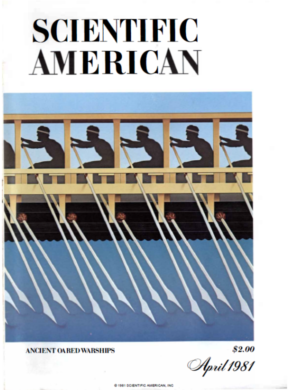

How to play?

- Given t colors, Alice and Bob take turns properly coloring an uncolored vertex.

- Alice: minimizer.
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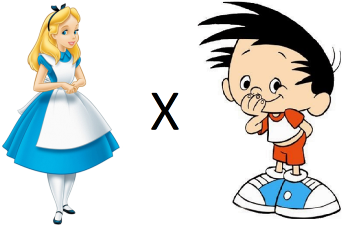

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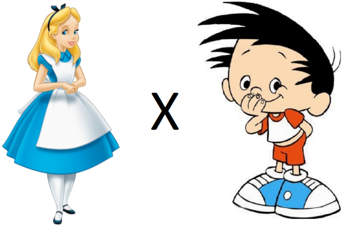

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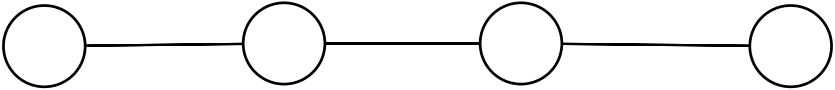

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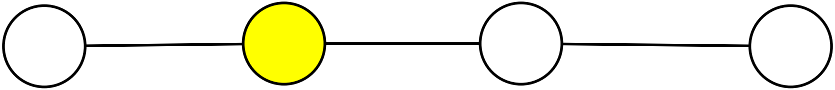

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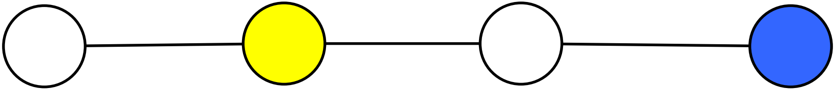

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Simple results

- $\chi(G) \leq \chi_g(G) \leq \Delta(G) + 1$
  - $\chi_g(K_n) = n$
  - $\chi_g(S_n) = 1$

- $\chi_g(P_1) = 1$, $\chi_g(P_2) = \chi_g(P_3) = 2$

- For $n \geq 4$, we have that $\chi_g(P_n) = 3$

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- The stars $K_{1,p}$ with $p \geq 1$ are the only connected graphs satisfying $\chi_g(G) = 2$

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Different graph classes studied

- planar graphs: $7 \leq \chi_g(P) \leq 17$;
- outerplanar graphs: $6 \leq \chi_g(O) \leq 7$;
- toroidal grids: $\chi_g(TG) = 5$;
- partial k-trees: $\chi_g(P) \leq 3k + 2$;
- the cartesian products of some classes of graphs: for example, $\chi_g(T_1 \Box T_2) \leq 12$;

Literature for trees

- Bodlaender (1991): $\chi_g(T) \leq 5$.
- Faigle et al. (1993): $\chi_g(F) \leq 4$.
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- Due to the difficulty concerning this subject, the problem of characterizing forests with $\chi_g(F) = 3$ remains open.

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- Example presented in Bodlaender (1991) to prove the existence of a tree H_d with $\chi_g(H_d) \geq 4$:

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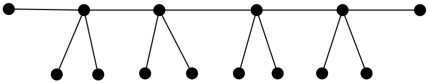

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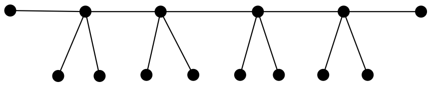

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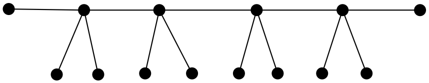

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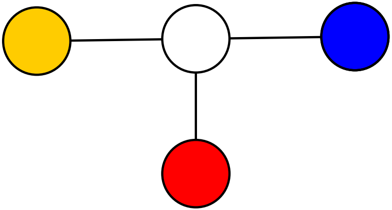
Claw



Our new parameters

- The game chromatic number is:
  - $\chi_g^a(G)$ (or simply $\chi_g(G)$): when Alice starts the game;
  - $\chi_g^b(G)$: when Bob starts the game!
  - $\chi_g(G, Z)$: when Alice starts the game in the partially colored graph G, for Z a set of vertices of $V(G)$ such that for all $v \in Z$, $c(v) \neq \emptyset$.

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Sufficient conditions for $\chi_g(H) = 4$ for any caterpillar H

Theorem

If a caterpillar H has an induced subcaterpillar H', such that $\chi_g^a(H') = \chi_g^b(H') = 4$, then $\chi_g^a(H) = \chi_g^b(H) = 4$.

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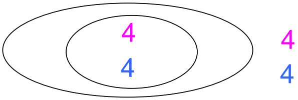

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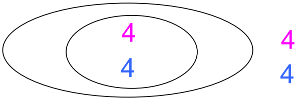

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If a caterpillar H has two induced subcaterpillars H' and H'', such that $\chi_g^b(H') = \chi_g^b(H'') = 4$, then $\chi_g^a(H) = \chi_g^b(H) = 4$.

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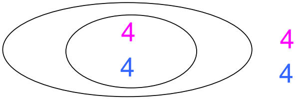

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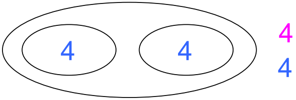

Necessary conditions for $\chi_g(H) = 4$ for any caterpillar H

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If a caterpillar H has $\chi_g(H) = 4$, then H has at least four vertices of degree at least 4.

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If H is a minimal caterpillar with respect to $\chi_g^a(H) = 4$, then H does not have consecutive vertices of degree 2, unless H has two edge disjoint induced subcaterpillars H' and H'' that are minimal with respect to $\chi_g^b(H') = 4$ and $\chi_g^b(H'') = 4$.

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Low degree vertices (degree 2) are important to have $\chi_g(H) \leq 3$.

Four infinite families

- Caterpillars
  - with maximum degree 3;
  - without vertex of degree 2;
  - without vertex of degree 3;
  - with vertices of degree 1, 2, 3 and 4.

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Let H be the caterpillar $cat(k_1, ..., k_s)$ with $\Delta(H) = 3$. We have that H has $\chi_g^a(H)$, $\chi_g^b(H) \leq 3$. Moreover, let F be the forest where each connected component is a caterpillar and $\Delta(F) = 3$. We have that F has $\chi_g^a(F) \leq 3$.

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Claim 1) If $Z = \{v_1, v_s\}$, then $\chi_g^a(H, Z)$, $\chi_g^b(H, Z) \leq 3$, **except for the caterpillars with s odd, which has $\chi_g^b(H, Z) \leq 4$.**

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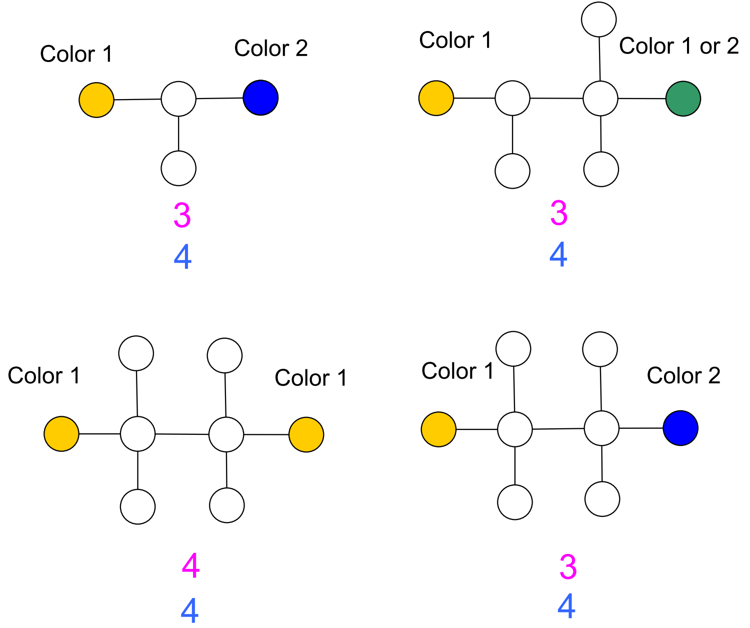
Two claws


Color 1
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Lemma (one vertex of degree at last 4)

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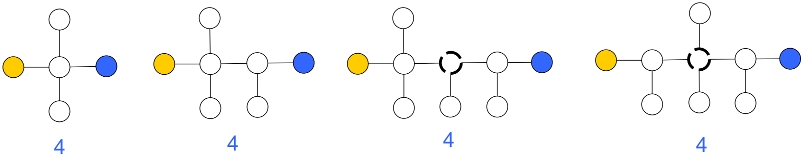

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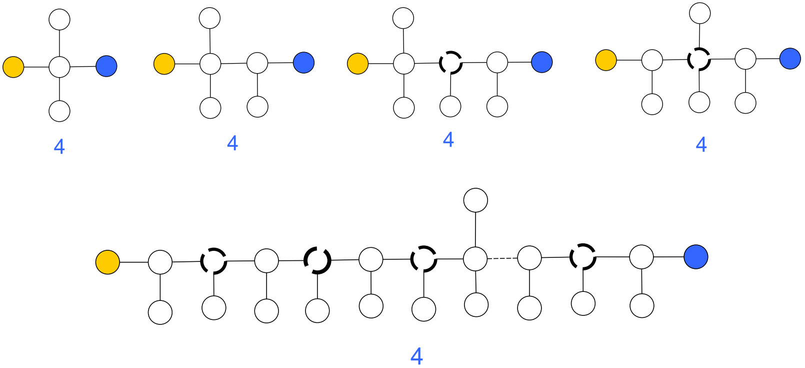

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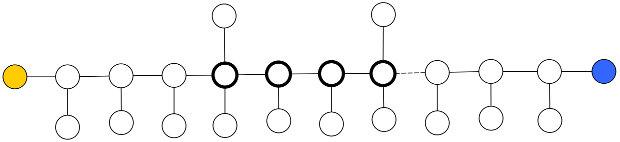

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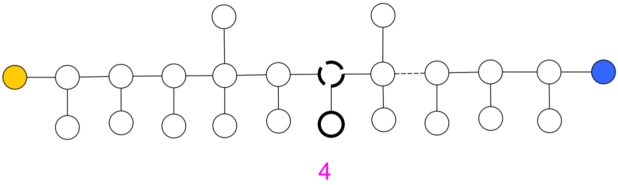

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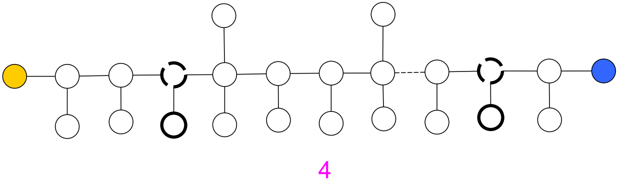

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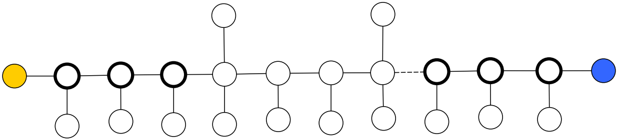

4

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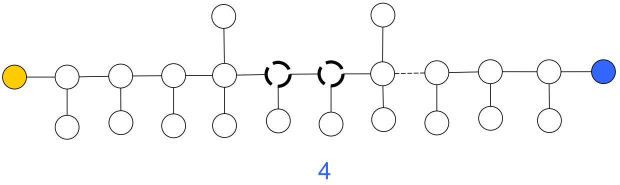

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Caterpillar without vertex of degree 2

Theorem

Let H be the caterpillar without vertex of degree 2. We have that $\chi_g^a(H) = \chi_g^b(H) = 4$ *if, and only if, H is caterpillar* $cat(k_1, ..., k_s)$*, such that* $k_1 = k_s = 0$, $k_i \neq 0$, $\forall i \in \{2, ..., s-1\}$*, and there are at least four vertices of degree at least 4.*

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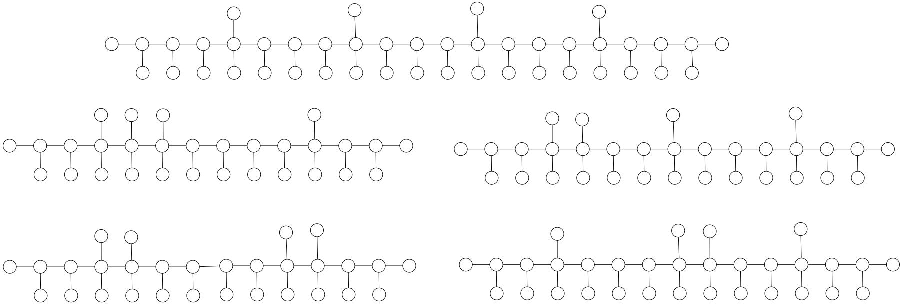
$\Rightarrow$ By the necessary condition for $\chi_g(H) = 4$.

$\Leftarrow$



Let *Family Q* be the set of caterpillars H_d, H_{33}, $H_{[\alpha]} \cup H_{[\beta]}$, $H_{[\alpha][\beta]}$ and $H_{[\alpha]3[\beta]}$.

Caterpillar without vertex of degree 3

Let *Family Q* be the set of caterpillars H_d, H_{33}, $H_{[\alpha]} \cup H_{[\beta]}$, $H_{[\alpha][\beta]}$ and $H_{[\alpha]3[\beta]}$.

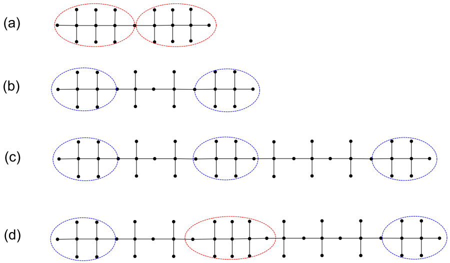


Figure: Caterpillars (a) H_{33} (b) $H_{[3]}$ (c) $H_{[3][4]}$ (d) $H_{[3]3[4]}$.

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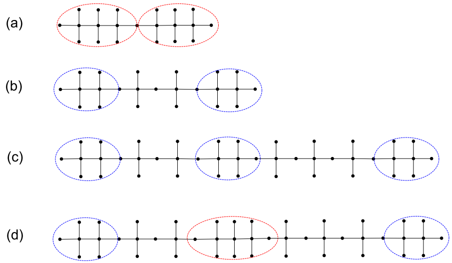

Figure: Caterpillars (a) H_{33} (b) $H_{[3]}$ (c)$H_{[3][4]}$ (d)$H_{[3]3[4]}$.

Theorem

A caterpillar H without vertex of degree 3 has $\chi_g(H) = 4$ if, and only if, H has a caterpillar of Family Q as an induced subcaterpillar.

Caterpillar with vertices of degree 1, 2, 3 and 4

Let *Family Q'* be the set of caterpillars $\{H'_{[\alpha]} \cup H'_{[\beta]},\ H'_{[\alpha]} \cup H_3,\ H_3 \cup H_3,$ H'_{22} and $H'_{[\alpha][\beta]}, H'_{23}\}$.

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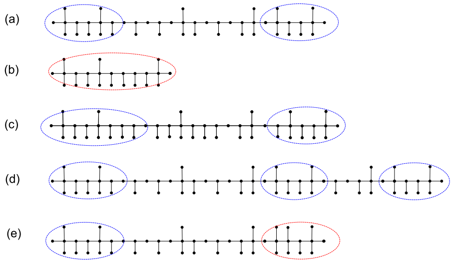

Figure: Caterpillars (a) $H'_{[6]}$ (b) H'_3 (c)H'_{22} (d)$H_{[6][3]}$ (e)H'_{23}.

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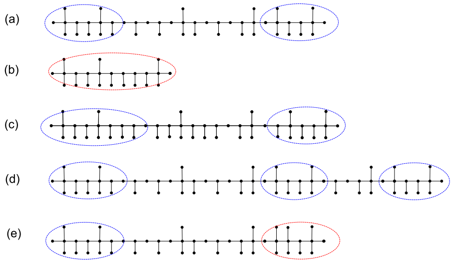

Figure: Caterpillars (a) $H'_{[6]}$ (b) H'_3 (c)H'_{22} (d)$H_{[6][3]}$ (e)H'_{23}.

Theorem

Let H be a caterpillar with vertices of 1, 2, 3 and 4. If H has a caterpillar of Family Q′ as a induced subcaterpillar, then $\chi_g(H) = 4$.

Summary

$\Delta(G)$	$\chi_g(G) = 1$	$\chi_g(G) = 2$	$\chi_g(G) = 3$	$\chi_g(G) = 4$
0	P_1	-	-	-
1	-	P_2	-	-
2	-	P_3	$P_n, n \geq 4$	-
3	-	star	not a star	-
4	-	star	see next Figure	see next Figure

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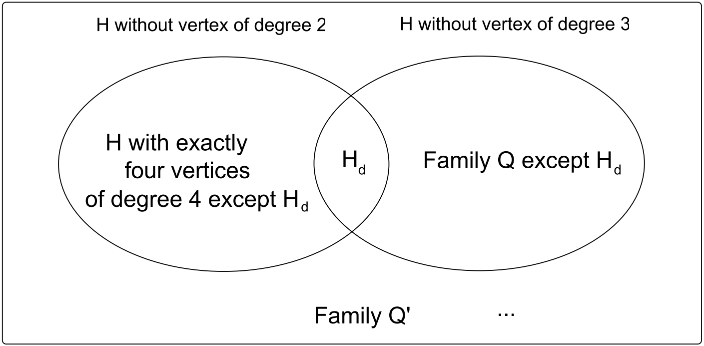

Figure: Caterpillars with $\Delta(H) = 4$ and $\chi_g(H) = 4$.

$\chi_g(F)$

Theorem

Let F be a forest composed by r trees T_1, ..., T_r. Assume that $\chi_g^a(T_1) \leq \chi_g^a(T_2) \leq ... \leq \chi_g^a(T_r)$, and, if there exist two trees with the same game chromatic number, then T_i and T_j are ordered in a way that $\chi_g^b(T_i) \leq \chi_g^b(T_j)$, for $i < j$. We have that:

1. *If $\chi_g^b(T_r) > \chi_g^a(T_r), \chi_g^b(T_{r-1})$, then $\chi_g(F) = \chi_g^a(T_r)$;*
2. *If $\chi_g^b(T_r) = \chi_g^b(T_{r-1}) > \chi_g^a(T_r)$, then $\chi_g(F) = \chi_g^b(T_r)$;*
3. *If $\chi_g^a(T_r) = \chi_g^b(T_r)$, then $\chi_g(F) = \chi_g^a(T_r) = \chi_g^b(T_r)$;*
4. *If $\chi_g^b(T_r) < \chi_g^a(T_r)$ and $\sum_{i=1}^{r-1} |V(T_i)|$ is even, then $\chi_g(F) = \chi_g^a(T_r)$;*
5. *If $\chi_g^b(T_r) < \chi_g^a(T_r)$ and $\sum_{i=1}^{r-1} |V(T_i)|$ is odd, then $\chi_g(F) = \max\left\{\chi_g^a(F \backslash T_r), \chi_g^b(T_r)\right\}$.*

Conclusion: Coloring Game

It is a reduction problem.

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It is a reduction problem.

We are able to characterize evil, indifferent and good subgraphs for Alice to win the game with 3 colors in caterpillars.

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Furtado, A., Dantas, S., Figueiredo, C., Gravier, S., Schimidt, S., *The Game Chromatic Number of Caterpillars.* In proceedings of the XVIII Latin-Iberoamerican Conference on Operations Research, Santiago (2016).

Nordhaus-Gaddum type inequalities

What are Nordhaus-Gaddum type inequalities?

- Nordhaus and Gaddum (1956) showed lower and upper bounds on the sum of the chromatic number of a graph and its complement:

- Survey by Aouiche and Hansen (2013): 360 articles.
- To the best of our knowledge, the only Nordhaus-Gaddum type inequality existing for invariants related to games on graphs is by Alon et al.(2002) and concerns the game domination number.

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Theorem (Nordhaus and Gaddum, 1956)

If G is a graph of order n, then $2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1$. These bounds are best possible for infinitely many values of n.

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Nordhaus-Gaddum type inequalities to $\chi_g(G) + \chi_g(\overline{G})$: Theorem 4.4

Theorem

Nordhaus and Gaddum For any graph G of order n, we have that $2\sqrt{n} \leq \chi_g(G) + \chi_g(\overline{G}) \leq \lceil \frac{3n}{2} \rceil$. Moreover, the bounds are best possible asymptotically:

1. *for infinitely many values of n, there are graphs G of order n with* $$\chi_g(G) + \chi_g(\overline{G}) = \left\lceil \frac{4n}{3} \right\rceil - 1;$$
2. *for infinitely many values of n, there are graphs G of order n with* $\chi_g(G) + \chi_g(\overline{G}) = 2\sqrt{2n} - 1$.

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The lower bound follows from Theorem of Nordhaus and Gaddum (1965) and the inequality $\chi(G) \leq \chi_g(G)$.

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Proof for the upper bound $\chi_g(G) + \chi_g(\overline{G}) \leq \lceil \frac{3n}{2} \rceil$

Case 1) n is even.

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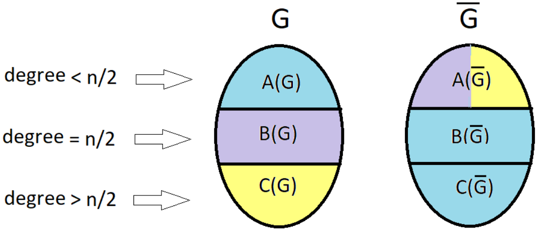

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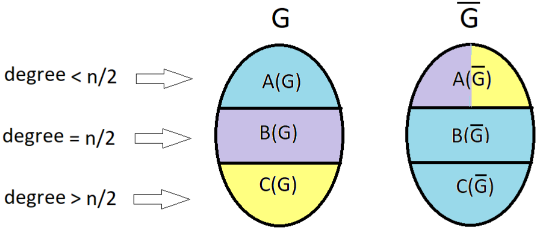

In G, Alice begins by coloring only in $B(G) \cup C(G)$ until those vertices are all colored. Assume that $\frac{n}{2}$ colors are used.

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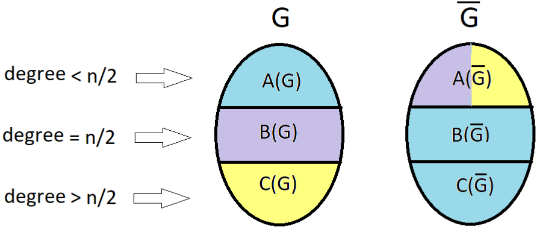

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Case 1.1) $b + c < \left\lceil \dfrac{n}{4} \right\rceil$.

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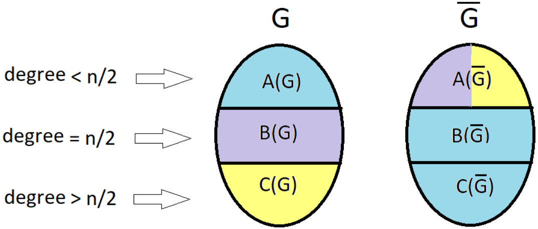

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Just vertices in $A(G)$ can be not colored and they do not need any different color, and $\chi_g(G) \leq \frac{n}{2}$.

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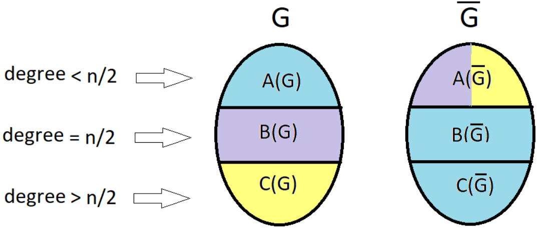

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As $\chi_g(\overline{G}) \leq n$, then $\chi_g(G) + \chi_g(\overline{G}) \leq \frac{3n}{2}$.

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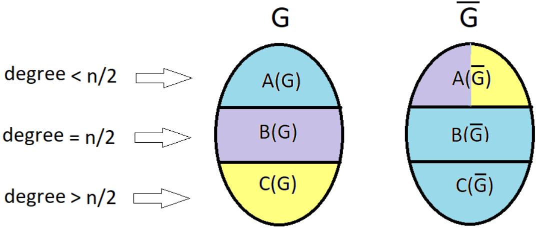

In G, Alice begins by coloring only in $B(G) \cup C(G)$ until those vertices are all colored. Assume that $\frac{n}{2}$ colors are used.

Case 1.2) $a < \left\lceil \frac{n}{4} \right\rceil$. As in case 1.1, $\chi_g(\overline{G}) \leq \frac{n}{2}$ and $\chi_g(G) \leq n$. So, $\chi_g(G) + \chi_g(\overline{G}) \leq \frac{3n}{2}$.

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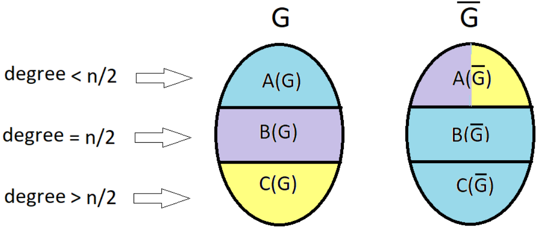

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Case 1.3) $a \geq \left\lceil \frac{n}{4} \right\rceil$ *and* $b + c \geq \left\lceil \frac{n}{4} \right\rceil$. There are at most $b + c - \left\lceil \frac{n}{4} \right\rceil$ uncolored vertices in $B(G) \cup C(G)$. If there are uncolored vertices in $A(G)$, they do not need any different color.

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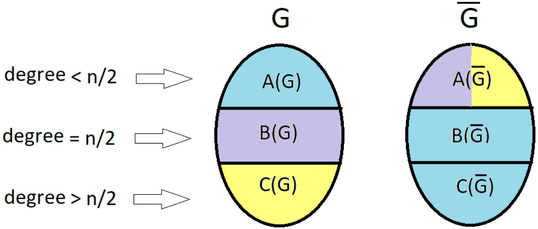

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So, $\chi_g(G) \leq \frac{n}{2} + b + c - \left\lceil \frac{n}{4} \right\rceil$.

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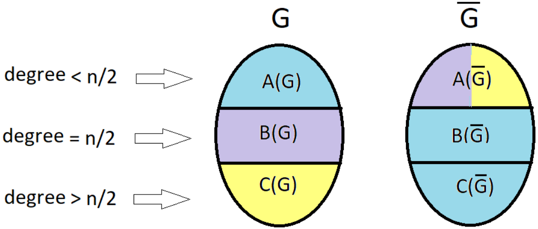

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Using exactly the same idea in $\overline{G}$, $\chi_g(\overline{G}) \leq \frac{n}{2} + a - \left\lceil \frac{n}{4} \right\rceil$.

Proof for the upper bound $\chi_g(G) + \chi_g(\overline{G}) \leq \lceil \frac{3n}{2} \rceil$

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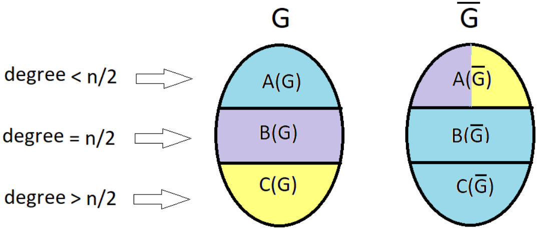

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$$\chi_g(G) + \chi_g(\overline{G}) \leq \frac{n}{2} + b + c - \left\lceil \frac{n}{4} \right\rceil + \frac{n}{2} + a - \left\lceil \frac{n}{4} \right\rceil \leq \frac{3n}{2}.$$

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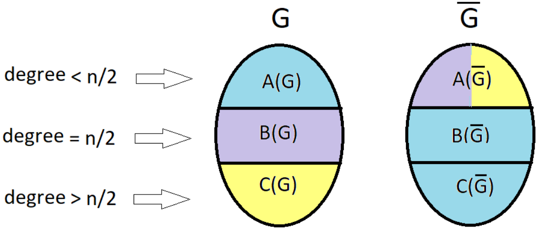

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Case 2) n is odd is similar and $\chi_g(G) + \chi_g(\overline{G}) \leq \left\lceil \frac{3n}{2} \right\rceil$.

Construction of family 1 of the Theorem 4.4

Lemma

Let G_l be the graph join $S_l \oplus K_{\lceil \frac{l}{2} \rceil}$, with order $n = l + \left\lceil \frac{l}{2} \right\rceil \not\equiv 1$ mod 3 and $n \geq 5$. We have that $\chi_g(G_l) + \chi_g(\overline{G_l}) = \left\lceil \frac{4n}{3} \right\rceil - 1$.

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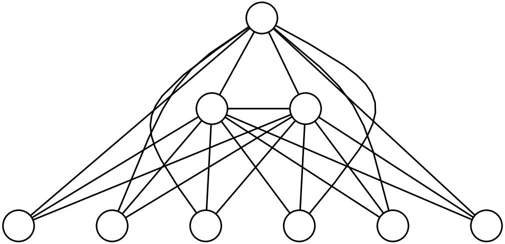

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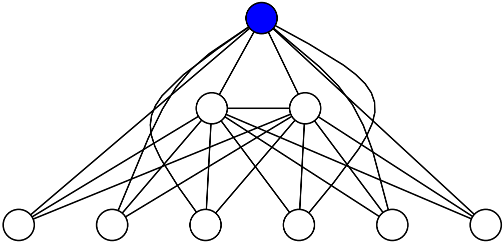

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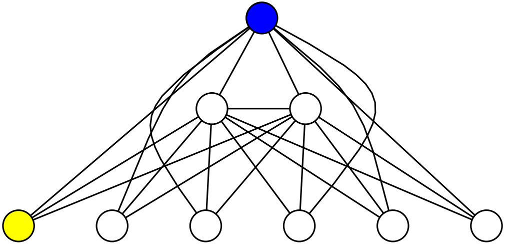

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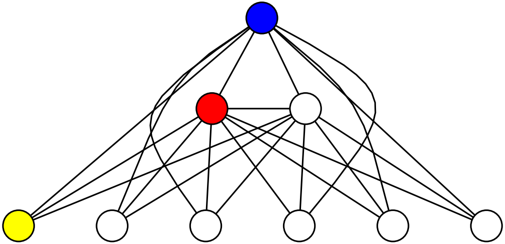
Let G_l be the graph join $S_l \oplus K_{\lceil \frac{l}{2} \rceil}$, with order $n = l + \left\lceil \frac{l}{2} \right\rceil \not\equiv 1 \mod 3$ and $n \geq 5$. We have that $\chi_g(G_l) + \chi_g(\overline{G_l}) = \left\lceil \frac{4n}{3} \right\rceil - 1$.

Alice colors first in the clique.



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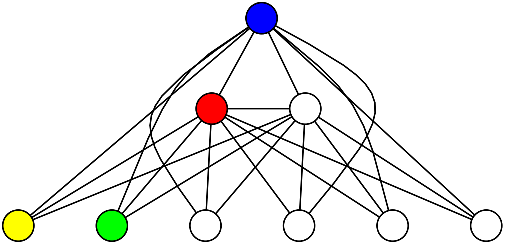

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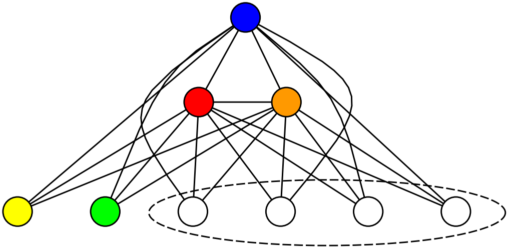

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When Alice finishes to color the vertices of the clique, she has played $\left\lceil \frac{l}{2} \right\rceil$ times, and Bob $\left\lceil \frac{l}{2} \right\rceil - 1$ times.

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$\overline{G_l}$ is composed by a clique K_l and a stable set $S_{\lceil \frac{l}{2} \rceil} \Rightarrow \chi_g(\overline{G_l}) = l$.

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Family 2 of the Theorem 4.4

Lemma

Let G be a complete $\sqrt{\frac{n}{2}}$-partite graph, such that $\sqrt{\frac{n}{2}}$ is an even integer and each $\sqrt{\frac{n}{2}}$ disjoint set of vertices has exactly $\sqrt{2n}$ vertices. We have that $\chi_g(G) + \chi_g(\overline{G}) = 2\sqrt{2n} - 1$.

Nordhaus-Gaddum type inequalities to other games

- We determine the Nordhaus-Gaddum type inequalities to
  - the number of P-positions of a caterpillar (Timber Game);
  - the *game coloring number* of any graph G (Marking Game).
- *Marking Game* is "colorblind" version of the coloring game.
- All bounds are tight, except the upper bound for the number of P-positions of a caterpillar.

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Submission

Charpentier, C., Furtado, A., Dantas, S., Figueiredo, C., Gravier, *On Nordhaus-Gaddum type inequalities for the Game Chromatic and Game Coloring numbers.* Submitted to Discrete Maths. (2018)

Current and Future work: Timber Game

Conjecture

The number of P-positions of family 1 is: $\dfrac{2(s-a+1)}{a-1}\dbinom{s-1}{(a-3)/2}$.

- Is there a simpler formula for the number of P-positions of family 3 without the use of summation?
- Given any caterpillar, is there a polynomial algorithm to determine its number of P-positions?

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Current and Future work: Coloring Game

Theorem

The caterpillar $H'_{[\alpha]}$ is the unique caterpillar with vertices of degree 1, 2, 3 and 4 satisfying $\chi_g^a(H'_{[\alpha]}) = 3$ and that is minimal with respect to $\chi_g^b(H'_{[\alpha]}) = 4$. ✓ *(LAWCG 2018)*

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If H is a caterpillar with vertices of degree 1, 2, 3 and 4, and is minimal with respect to $\chi_g(H) = 4$, then H is a caterpillar of Family Q'.

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Conjecture

$\chi_g^a(T) \leq \chi_g^b(T)$, *for a tree T, except for $T = P_4$.*

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Current and Future work: Nordhauss-Gaddum

- Is it possible to improve the upper bound for the number of P-positions in a caterpillar so that the bound is tight?

- Is it possible to find extremal graphs for the lower and upper bounds for the number of P-positions in a caterpillar, the game chromatic and coloring numbers in any graph?

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Current and Future work in general

- Apply the games in high school, college classes and events of the popularization of mathematics.

THANK YOU!