Sufficient conditions for hyper-Hamiltonicity in graphs

Cybele T. M. Vinagre
Instituto de Matemática e Estatística- UFF- Niterói
cybl@vm.uff.br

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Joint work with Renata Del-Vecchio and Guilherme B. Pereira

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1. Introduction

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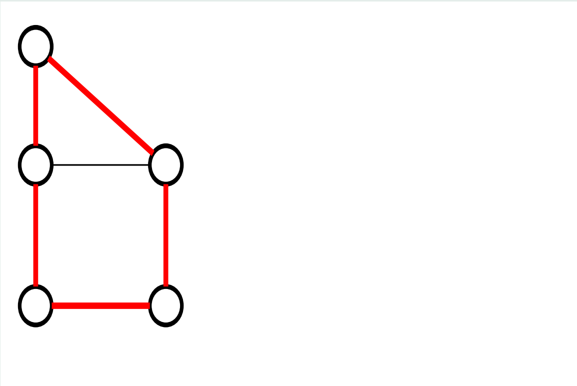

Figure 1: An Hamiltonian graph with an hamiltonian cycle.

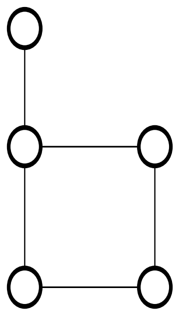

Figure 2: A non Hamiltonian graph.

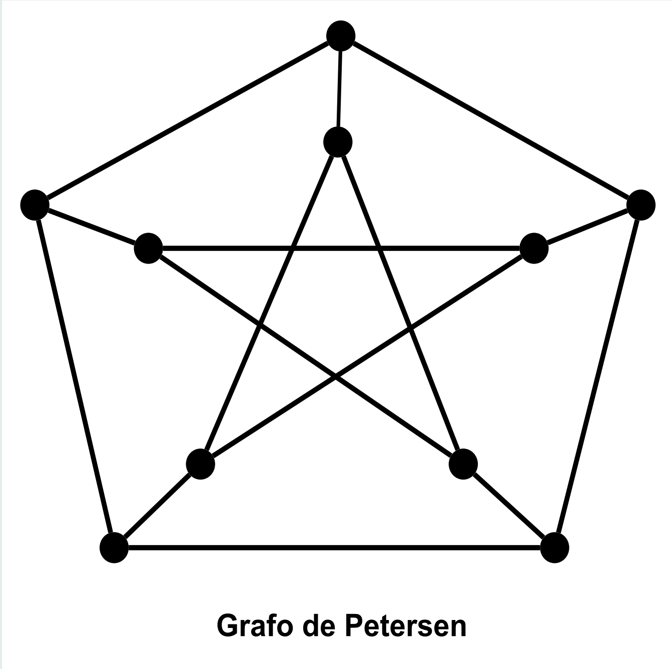

Figure 3

A graph G is said to be hyper-Hamiltonian when G is Hamiltonian and $G - \{v\}$ is also Hamiltonian for any vertex v of G.

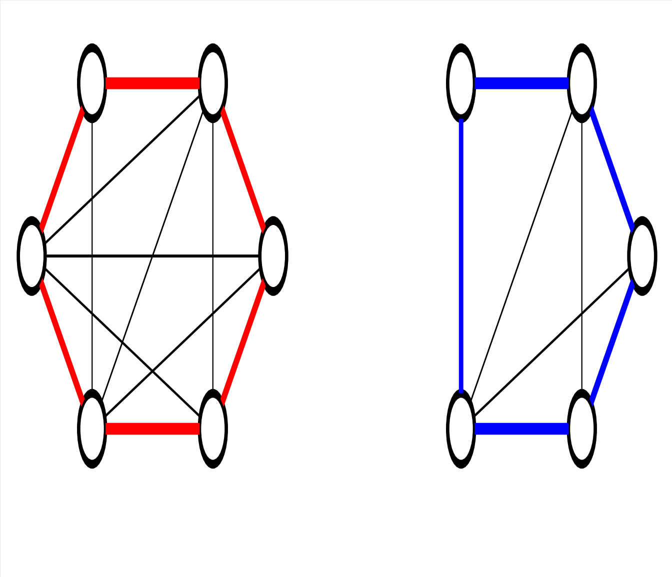

Figure 4: An hiper-Hamiltonian graph

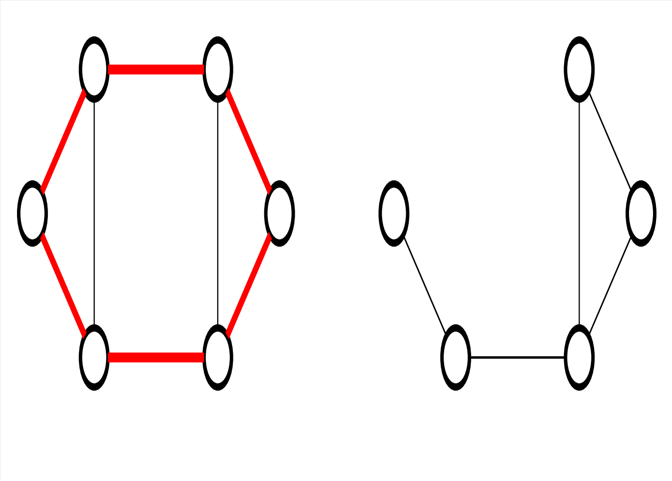

Figure 5: A non hiper-Hamiltonian graph

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In this work, we present some sufficient conditions to ensure that an arbitrary graph is hyper-Hamiltonian, in analogy to results on Hamiltonicity.

We hope, this way, be providing the basis for future research on the topic.

2. General conditions for hyper - Hamiltonian graphs

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$$d_G(u) + d_G(v) \geq n \text{ for every pair of nonadjacent vertices } u \text{ and } v$$

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Our first theorem is an analogous to Ore's theorem for hyper-Hamiltonian graphs.

Theorem 2 *Let* G *be a graph with* $n \geq 3$ *vertices. If*

$$d_G(u) + d_G(v) \geq n + 1 \quad \textit{for every pair of nonadjacent vertices } u \textit{ and } v$$

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then G is hyper-Hamiltonian.

Sketch of the proof: It is enough to apply Ore's theorem to $G' = G - \{w\}$, considering the three possibilities on vertices u, v and w:

- ◇ G has the edges uw and vw;
- ◇ the edges uw and vw are not in G;
- ◇ G has the edge uw but not the edge vw.

As an immediate consequence we also have an analogous to Dirac's theorem [5] (1952).

Corollary 3 *If* $\delta(G) \geq \dfrac{n+1}{2}$ *then* G *is hyper-Hamiltonian.* *(*$\delta(G)$ *denotes the minimum degree among vertices of* G*)*

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◇ $\mathbb{P}_n + e$ the graph obtained from $\mathbb{P}_n$ by inserting an edge.

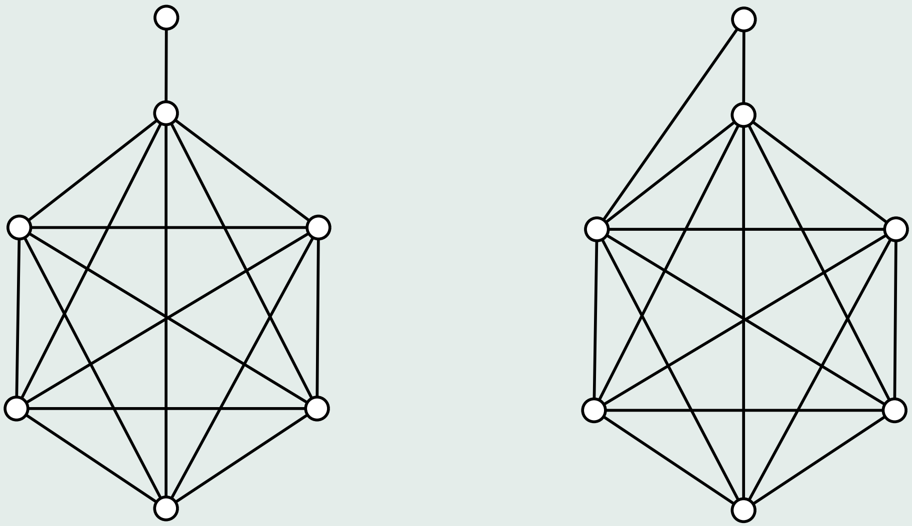

Figure 6: $\mathbb{P}_6$ **and** $\mathbb{P}_6 + e$

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Theorem 4 *Let G be a graph with $n \geq 3$ vertices and m edges.*

If $m > \dfrac{n^2 - 3n + 4}{2}$ then G is hyper-Hamiltonian or $G = \mathbb{P}_{n-1} + e$.

Theorem 5 *Let G be a graph with $n \geq 3$ vertices and m edges.*

If $m \geq \dfrac{n^2 - 3n + 6}{2}$ then G is hyper-Hamiltonian unless $G = \mathbb{P}_{n-1} + e$.

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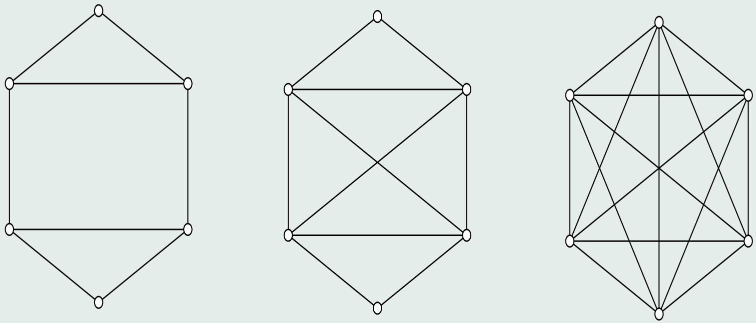

Figure 7: A graph and its 3-closure.

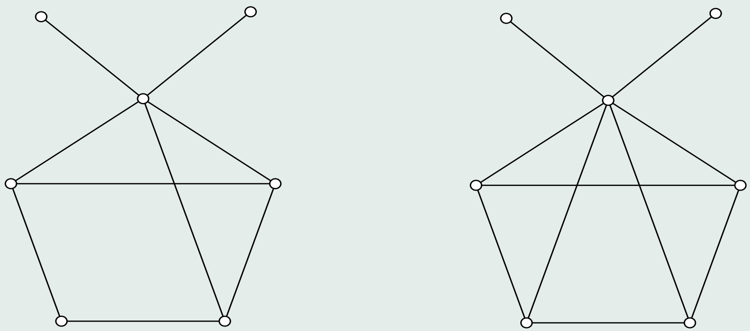

Figure 8: A graph and its 7-closure.

The k-closure of a graph allows to state the following proposition, analogous to one found in [2].

Proposition. 7 *A graph G on n vertices is hyper-Hamiltonian if, and only if, the $(n+1)$-closure of G is hyper-Hamiltonian.*

Definition 8 *Let P be a property defined for all graphs of order n and k be a nonnegative integer. We say that P is a k-stable property if for all pairs of non adjacent vertices u and v in a graph G of n order, whenever $d_G(u) + d_G(v) \geq k$ and $G + uv$ has property P and then G must have property P.*

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In [2], it was proved for P a property defined for all graphs of order n:

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Proposition. 9 *The property of being hyper-Hamiltonian is $(n + 1)$-stable.*

3. Spectral conditions for hyper-Hamiltonicity

3.1. Spectral conditions for hyper-Ham on spectral radius of adjacency matrix

The *adjacency matrix* of G, $\mathbf{A} = [a_{ij}]$, is the $n \times n$ matrix for which the entries are $a_{ij} = 1$ if $ij \in E(G)$, and 0 otherwise.

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Let $\lambda(G)$ denote the spectral radius of the adjacency matrix of a graph G, i.e., its largest eigenvalue.

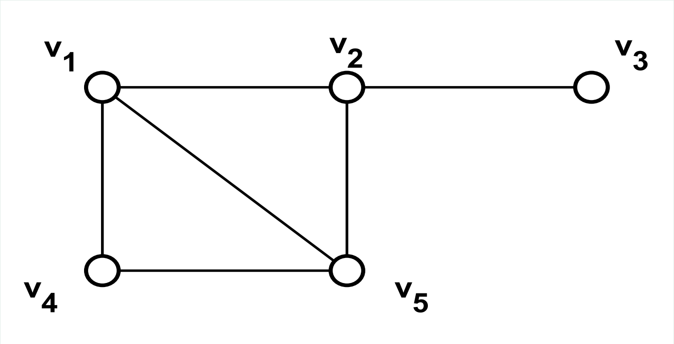

Figure 9: Graph G.

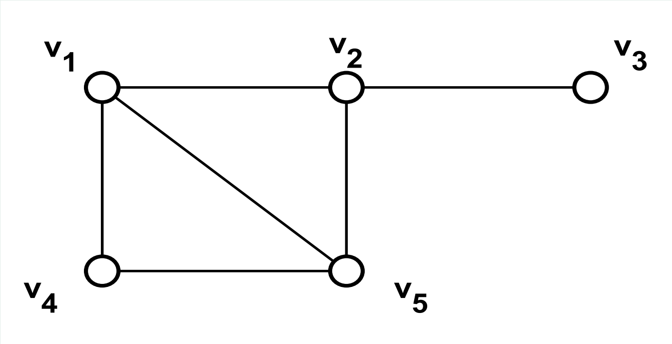

Figure 9: Graph G.

The adjacency matrix of G is

$$A(G) = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

$$spect(G) = \begin{bmatrix} 2,6412 & 0,7237 & -0,5892 & -1 & -1,7757 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Thus $\lambda(G) = 2,6412$.

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Theorem 10 *[7] Let G be a graph with $n \geq 3$ vertices. If $\lambda(G) > n-2$ then G is Hamiltonian unless $G = \mathbb{P}_{n-1}$.*

If $\lambda(\overline{G}) < \sqrt{n-2}$ then G is Hamiltonian unless $G = \mathbb{P}_{n-1}$.

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These results motivated many other spectral conditions for Hamiltonicity, as in [14] and [13], for instance.

Theorem 11 *Let G be a graph with n vertices. If $\lambda(G) > -\frac{1}{2} + \sqrt{\left(n - \frac{3}{2}\right)^2 + 2}$ then G is hyper-Hamiltonian or $G = \mathbb{P}_{n-1} + e$.*

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Stanley's inequality ($\lambda(G) \leq -\frac{1}{2} + \sqrt{2m + \frac{1}{4}}$, where m is the number of edges in G), furnishes thus

$$-\frac{1}{2} + \sqrt{\left(n - \frac{3}{2}\right)^2 + 2} < \lambda(G) \leq -\frac{1}{2} + \sqrt{2m + \frac{1}{4}}$$

which implies $m \geq \dfrac{n^2 - 3n + 6}{2}$, which allows the use of Theorem 5, concluding the proof.

Theorem 12 *Let G be a graph with n vertices and $\lambda(\overline{G})$ be the spectral radius of its complement $\overline{G}$. If* $\lambda(\overline{G}) \leq \sqrt{\binom{n-2}{2} - \binom{n-2}{n}}$ *then G is hyper-Hamiltonian or $G = \mathbb{P}_{n-1} + e$.*

Sketch of the proof: Assuming that G is not hyper-Hamiltonian, Proposition 7 tells us that its $n+1$-closure I is not either.

Furthermore, for every pair of nonadjacent vertices u and v of I, $d_I(u)+d_I(v) \leq (n+1)-1 = n$.

Turning to the complement $\overline{I}$ and applying Hofmeister's inequality

($\lambda \geq \sqrt{\frac{1}{n}(d^2(v_1) + \ldots + d^2(v_n))}$)

to $\overline{I}$, we achieve a contradiction.

3.2. Spectral conditions for hyper-Ham on spectral radius of matrices Q and D

Let $Deg(G)$ be the diagonal matrix whose (i,i)-entry is the degree of vertex v_i and $A(G)$ the adjacency matrix of G. The matrix $Q(G) = Deg(G) + A(G)$ is the signless Laplacian matrix of G.

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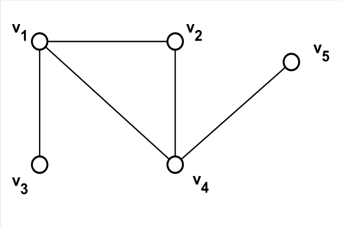

Figure 10: Graph

$$Deg(G) = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A(G) = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Thus

$$Q(G) = Deg(G) + A(G) = \begin{bmatrix} 3 & 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

The signless Laplacian spectral radius of G is the largest eigenvalue of $Q(G)$, denoted by $q_1(G)$.

Similar to what is done in [14], we obtain a condition for hyper-Hamiltonicity, based on this parameter.

Consider the set of graphs on n vertices

$$\mathcal{E}_n = \{G : G = P_2 \vee (K_a \cup K_{n-a-2}), a \in \mathbb{N}, 1 < a+2 < n\} \cup$$

$$\{G : G \text{ is bipartite and } n/2\text{-regular}\} \cup$$

$$\left\{G = H \vee F : H \text{ is } \left(\frac{n}{2} - r\right)\text{-regular and } \mid F \mid = r < \frac{n}{2}\right\}$$

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where $\vee$ indicates the join operation.

Theorem 13 *Let G be a graph with n vertices, for $n \geq 3$.*
If $q_1(\overline{G}) \leq n-2$ and $G \notin \mathcal{E}_n$ then G is hyper-Hamiltonian.

Let $D(G)$ be the distance matrix of a connected graph G, that is, the matrix whose (i, j)-entry is $d(v_i, v_j)$, the distance between vertices v_i and v_j.

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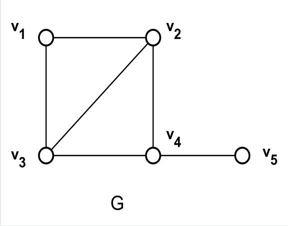

Figure 11: G.

Exemplo 14

$$D(G) = \begin{bmatrix} 0 & 1 & 1 & 2 & 3 \\ 1 & 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 0 & 1 \\ 3 & 2 & 2 & 1 & 0 \end{bmatrix}.$$

We denote by $\rho(G)$ the spectral radius of $D(G)$ (largest eigenvalue of $D(G)$). The graph in example has $\rho(G) = 6,2161$.

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Theorem 15 *Let G be a connected graph with $n \geq 4$ vertices.*
If $\rho(G) < \dfrac{(n-1)(n+2)-2}{n}$ *then G is hyper-Hamiltonian or $G = \mathbb{P}_{n-1} + e$.*

Theorem 16 *Let G be a connected graph with $n \geq 4$ vertices, such that $\overline{G}$ is connected.*
If $\rho(\overline{G}) > n - \frac{5}{2} + 3\sqrt{\left(n - \frac{3}{2}\right)^2 + 2}$ then G is hyper-Hamiltonian or $G = \mathbb{P}_{n-1} + e$.

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Recall that the Laplacian matrix of G is given by $L(G) = Deg(G) - A(G)$.

We shall denote the eigenvalues of $L(G)$ in non increasing order as

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0.$$

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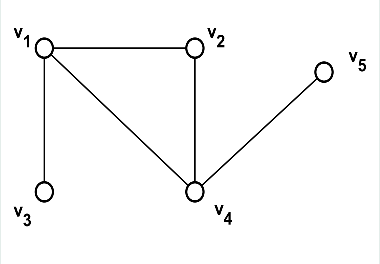

Figure 12: Graph

$$D = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Thus

$$L(G) = Deg(G) + A(G) = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

In this section, unlike what was done previously, we will restrict our results to a specific class of graphs, namely, threshold graphs.

Threshold graphs are graphs free of P_4, C_4 and $2K_2$ [4].

Hamiltonicity in threshold graphs is studied in [6] under a non spectral approach.

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Sketch of proof: Supose $\mu_{n-1} + \mu_{n-2} \geq n+1$.

Let $d_1 \geq d_2 \geq \ldots \geq d_n$ be the degree sequence of vertices of G.

From Merris' result,

$\mu_{n-1} = d_n$ and $\mu_{n-2} = d_{n-1}$.

Since d_n and d_{n-1} are the two smallest degrees among vertices of G, we have that for each pair of non adjacent vertices v_1, v_2 of G

$$d_G(v_1) + d_G(v_2) \geq d_n + d_{n-1} = \mu_{n-1} + \mu_{n-2} \geq n+1.$$

From Theorem 2, G is hyper- Hamiltonian.

An immediate consequence is the next corollary.

Corollary 18 *Let G be a threshold graph with n vertices. If $a(G) \geq \frac{n+1}{2}$ then G is hyper-Hamiltonian.*

We may note that different matrices do not produce the same conclusion considering hyperhmiltonicity of graphs as can be seen in the following example:

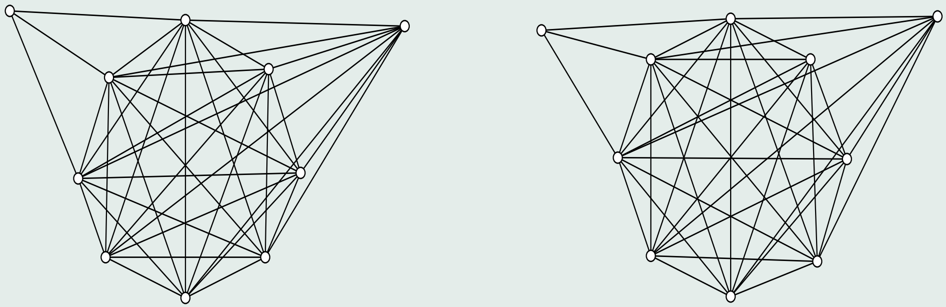
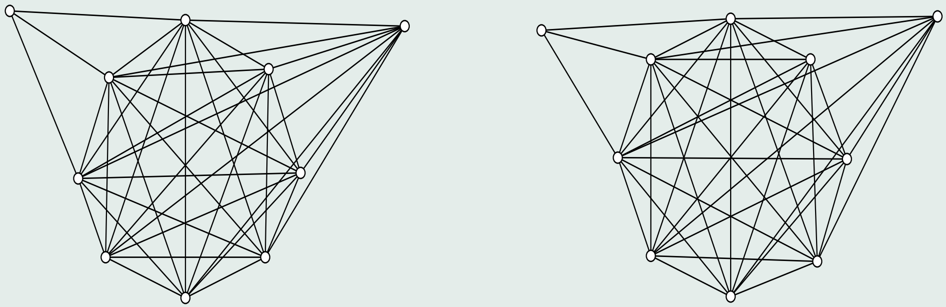


Figure 13: Two hyper-Hamiltonian graphs of Example 19.

Exemplo 19 *Both hyper-Hamiltonian graphs in Figure 2 have 10 vertices and non connected complements.*



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The graph G_1 *on the left has* $m = 39$, $\lambda(G_1) = 8,126$, $\lambda(\overline{G_1}) = 2,44$, $q_1(\overline{G_1}) = 7$, $\rho(G_1) = 10,43$, $\mu_{n-1}(G_1) = 3$ *and* $\mu_{n-2}(G_1) = 9$.

The graph G_1 *satisfies the conditions of Theorems 5, 11, 13, 15 and 17, but it does not satisfy conditions of Theorem 12 nor Corollary 18.*

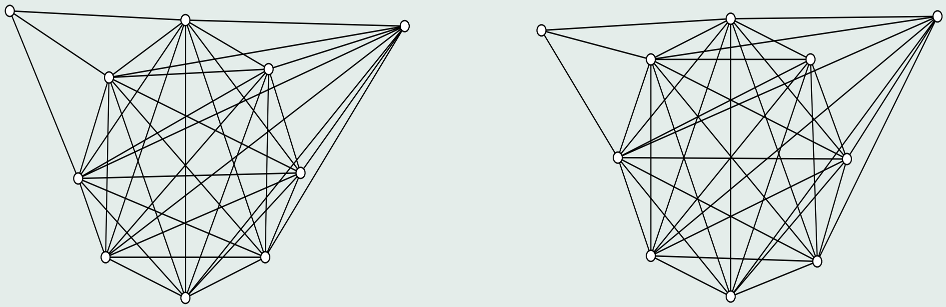

Figure 13: Two hyper-Hamiltonian graphs of Example 19.

Exemplo 19 *Both hyper-Hamiltonian graphs in Figure* 2 *have 10 vertices and non connected complements.*

The graph G_1 *on the left has* $m = 39$, $\lambda(G_1) = 8,126$, $\lambda(\overline{G_1}) = 2,44$, $q_1(\overline{G_1}) = 7$, $\rho(G_1) = 10,43$, $\mu_{n-1}(G_1) = 3$ *and* $\mu_{n-2}(G_1) = 9$.

The graph G_1 *satisfies the conditions of Theorems 5, 11, 13, 15 and 17, but it does not satisfy conditions of Theorem 12 nor Corollary 18.*

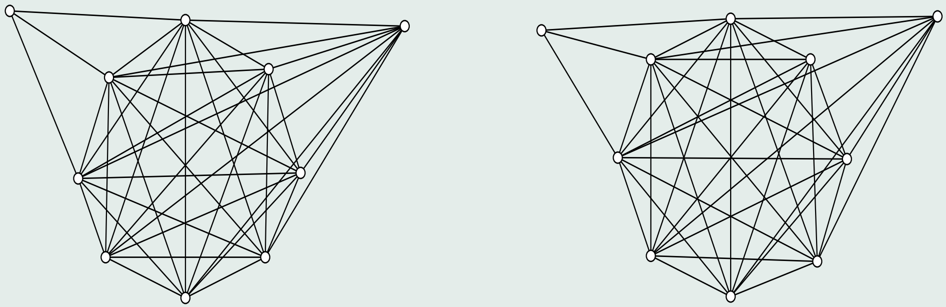

Figure 14: Two hyper-Hamiltonian graphs of Example 19.

Exemplo 20 *The graph* G_2 *on the left has* $m = 38$, $\lambda(G_2) = 7,93$, $\lambda(\overline{G_2}) = 2,68$, $q_1(\overline{G_2}) = 7,13$, $\rho(G_2) = 10,64$, $\mu_{n-1}(G_2) = 3$ *and* $\mu_{n-2}(G_2) = 7$.

Therefore, G_2 *satisfies Theorems 5 and 13 but do not satisfies Theorems 11, 12, 15 or 17.*

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