Relating hypergraph parameters of generalized power graphs

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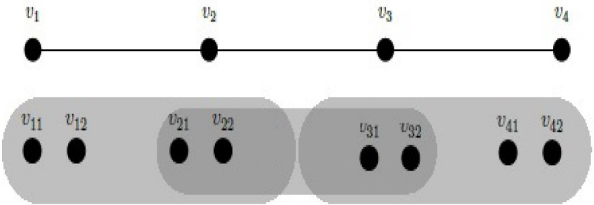
Introduction

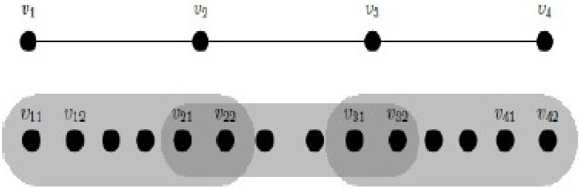
- A **hypergraph** $H = (V, E)$ is given by a vertex set V and a set $E = \{e : e \subseteq V\}$.
- H is **k-uniform** if $|e| = k$ for every edge $e \in E$.
- A hypergraph is **simple** if it has no loops (edges with $|e| = 1$) and if given any pair of edges, no edge contains the other.

Let G be a graph and $s \geq 1$ an integer.
The **s-extension** G_s of G is a $2s$-uniform hypergraph obtained from G by replacing each vertex $v_i \in V$ by a set $S_{v_i} = \{v_{i1}, \ldots, v_{is}\}$, where $S_{v_i} \cap S_{v_j} = \emptyset$ for every $v_i \neq v_j$.



More precisely, $V(G_S) = \{v_{11}, \ldots, v_{1s}, \ldots, v_{n1}, \ldots, v_{ns}\}$ and $E(G_s) = \left\{S_{v_i} \cup S_{v_j} : \{v_i, v_j\} \in E\right\}$.
Note that $|V(G_s)| = s \cdot |V(G)|$ and $|E(G_s)| = |E(G)|$.

Let $s \geq 1$ and $k \geq 2s$ be two integers and consider a graph G. The **generalized power graph** G_s^k is the k-uniform hypergraph $(G_s)^k$, obtained by adding $k - 2s$ new vertices to each edge of G_s, called **additional vertices**.



Note that $|V(G_s^k)| = s \cdot |V(G)| + (k - 2s) \cdot |E(G)|$ and $|E(G_s^k)| = |E(G)|$.

- When $s = 1$ we have that $G_s = G$.
- When $k = 2s$ we have that $G_s^k = G_s$
- The case where $G_s^k = G$ (ie, $s = 1$ and $k = 2s$) will not be considered.
- If G is a simple graph then G_s^k is a simple hypergraph. We will always consider G simple and with at least one edge.

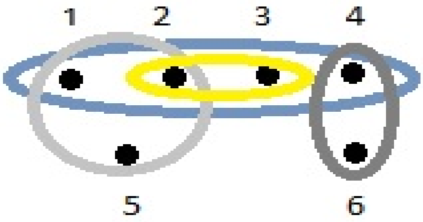
Let H be a hypergraph with n vertices.

The **adjacency matrix** of H, denoted by $A(H)$ is the $n \times n$ symmetric matrix:

$$a_{ij} = |\{e \in E(H) : v_i, v_j \in e\}|.$$

We denote the eigenvalues of $A(H)$ as $\lambda_1(H) \geq \ldots \geq \lambda_n(H)$.

Example:



$$A(H) = \begin{bmatrix} 0 & 2 & 1 & 1 & 1 & 0 \\ 2 & 0 & 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

We study

- Graph parameters can be generalized to hypergraphs and most of them in more than one way. The behavior of these parameters on the class G_s^k and their relation with the respective parameters of the original graph G.
- Spectral properties given by the adjacency matrix and establish relations with structural parameters of the hypergraph.

Diameter

A **path** P in a hypergraph H is a vertex-edge alternating sequence: $P = v_0,\ e_1,\ v_1,\ e_2,\ \ldots,\ v_{r-1},\ e_r,\ v_r$ such that $v_0, v_1, \ldots, v_r$ are distinct vertices; $e_1, e_2, \ldots, e_r$ are distinct edges; and $v_{i-1}, v_i \in e_i$, $i = 1, 2, \ldots, r$. The **length** of a path P is the number of distinct edges. A hypergraph is **connected** if for any pair of vertices, there is a path which connects them.

The **distance** $d(v, u)$ between two vertices v and u of a connected hypergraph is the minimum length of a path that connects v and u. The **diameter** $d(H)$ of H is defined by $d(H) = max\,\{d(v, u) : v, u \in V\}$.

- If G is connected if and only if G_s^k is connected.
- $d(G_s) = d(G)$.

Proposition

$d(G) \leq d(G_s^k) \leq d(G) + 2$, *for any connected graph* G.

Sketch of proof: By definition, $d(G_s) \leq d(G_s^k)$.

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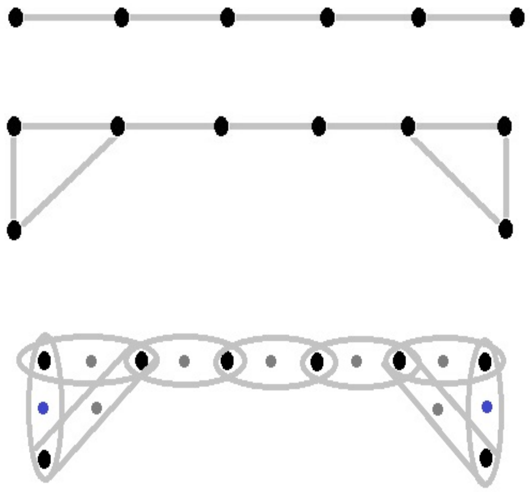
To increase the diameter from G_s to G_s^k we need to use the additional vertices. Notice that we can only have 2 additional vertices on a path and the path must start and end on them. Hence, $d(G_s^k) \leq d(G) + 2$.



Hypergraph Coloring

A **hypergraph coloring** is an assigning of colors $\{1, 2, \ldots, c\}$ to each vertex of $V(H)$ in such a way that each edge contains at least two vertices of distinct colors.

A coloring using at most c colors is called a c-coloring.

The *chromatic number* $\chi(H)$ of a hypergraph H is the least integer c such that H has a c-coloring.

- Given a graph G we have that $\chi(G_s) = \chi(G_s^k) = 2$.
- A hypergraph $H = (V, E)$ is **bipartite** if its vertex set V can be partitioned into two sets, X and Y such that for every edge $e \in E$, $e \cap X \neq \emptyset$ and $e \cap Y \neq \emptyset$.
- A hypergraph H is bipartite if and only if H is $2-$colorable.
- G_s^k is bipartite for any graph G.

Another type of coloring (also a generalization of graph coloring):

Strong hypergraph coloring is an assigning of colors $\{1, 2, \ldots, c\}$ to each vertex of $V(H)$ in such a way that every vertex of an edge has distinct colors.

The *strong chromatic number* $\chi_S(H)$ of a hypergraph H is the least integer c such that H has a strongly c-coloring.

- $\chi_S(H) \geq |e|$ for every $e \in E(H)$;
- $\chi(H) \leq \chi_S(H)$, since a strong hypergraph coloring is also a hypergraph coloring;
- Let H be a k-uniform hypergraph. A set $U \subseteq V(H)$ is a **clique** if every subset of U with k elements is an edge of H. The **clique number** is $\omega(H) = max\{|U| : U \subseteq V(H)$ is a clique$\}$;
- $\omega(H) \leq \chi_S(H)$ (similarly to graphs);
- The inequality $\omega(H) \leq \chi(H)$ is not valid, since $\chi(G_s^k) = 2$ but we can have edges (cliques) arbitrarily large.

Proposition

If G is a graph then $\chi_S(G_s) \leq s.\chi(G)$.

Proof: Let $\chi(G) = c$, we obtain a sc-strong coloring of G_s as follows:
if $v \in V(G)$ has color $x \in \{1, \ldots, c\}$ then, in G_s, assign colors $\{1 + (x-1)s, 2 + (x-1)s, \ldots, s + (x-1)s\}$ to S_v. □

Proposition

Let $s \geq 1$, $k > 2s$ be two integers and let G be a graph. We have that:
(i) if $\chi_S(G_s) < k$ then $\chi_S(G_s^k) = k$;
(ii) if $\chi_S(G_s) \geq k$ then $\chi_S(G_s^k) = \chi_S(G_s)$.

Proof: **(ii)** Let $\chi_S(G_s) = c \geq k$ and consider a c-strong coloring of G_s. For each edge of G_s^k, we color the $2s$ vertices that came from G_s with the same subset of $\{1, 2, .., c\}$ used in G_s, and we color the $k - 2s$ additional vertices with any $k - 2s$ distinct colors from $\{1, 2, .., c\}$, different from the $2s$ colors already used (since $c - 2s \geq k - 2s > 0$ such colors exist).

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Suppose that it is possible to use less than c-colors in G_s^k. This implies that we can color all the vertices of G_s^k that came from G_s with less than c-colors and hence G_s with less than c colors, a contradiction.

Independence Number

A set $U \subseteq V$ is an **independent set** if no edge of H is contained in U.

The **independence number** is $\alpha(H) = max\{|U| : U \subseteq V(H)$ is an independent set of H$\}$.

Proposition

Let G be a graph, then
$\alpha(G_s^k) \geq (s-1) \cdot |V(G)| + \alpha(G) + (k-2s) \cdot |E(G)|$.

Proof: Let $V(G) = \{v_1, \ldots, v_n\}$ and $V(G_s) = S_{v_1} \cup \ldots \cup S_{v_n}$.

We obtain an independent set with $(s-1) \cdot n$ elements by choosing $s-1$ vertices of S_{v_i}, for each $i = 1, \ldots, n$.

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Now, adding a maximum independent set of G to the previous set produces an independent set of G_s with $(s-1) \cdot n + \alpha(G)$ vertices.

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This independent set is also an independent set of G_s^k. Adding to it every $k-2s$ additional vertices of each edge of G_s^k produces an independent set of G_s^k with
$(s-1) \cdot |V(G)| + \alpha(G) + (k-2) \cdot |E(G)|$ elements. □

Adjacency Matrix of G_s

We managed to write the adjacency matrix of G_s in terms of graph matrices. For that we need some definitions.

Let G be a graph with n vertices.

- **Adjacency matrix** of G: $A(G)$ with entries $a_{ij} = 1$ if v_i and v_j are adjacent; and $a_{ij} = 0$ otherwise.
- **Degree matrix** of G: $D(G)$ is the diagonal matrix whose entries are the vertex degrees of G;
- **Signless Laplacian matrix of** G : $Q(G) = D(G) + A(G)$.

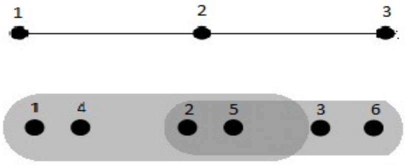
We denote the eigenvalues of $A(G)$ as $\lambda_1(G) \geq \ldots \geq \lambda_n(G)$ and the eigenvalues of $Q(G)$ as $q_1(G) \geq \ldots \geq q_n(G)$.

Proposition

Let G be a graph with n vertices. The adjacency matrix $A(G_s)$ is given on $s \times s$ blocks of size $n \times n$ by:

$$A(G_s) = \begin{bmatrix} A(G) & Q(G) & Q(G) & \dots & Q(G) \\ Q(G) & A(G) & Q(G) & \dots & Q(G) \\ Q(G) & Q(G) & A(G) & \dots & Q(G) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Q(G) & Q(G) & Q(G) & \dots & A(G) \end{bmatrix}.$$

Sketch of proof:



If we label the vertices like in the example above, we can see that the entries $a_{14} = d(1)$, $a_{25} = d(2)$ and $a_{36} = d(3)$. Every other entry of $A(G_s)$ is the same as the respective entry on $A(G)$. □

Proposition

Let G be a graph on n vertices and $d_1, \ldots, d_n$ the vertices degrees of G. Then $-d_1, \ldots, -d_n$ are eigenvalues of $A(G_s)$. Moreover, each $-d_i$ has multiplicity at least $s-1$.

Proof: Consider the vector v such that:
$v = (-1, 0, \ldots, 0|, 1, 0, \ldots, 0|, 0, \ldots, 0|, \ldots, |0, \ldots, 0) \in R^{sn}$, formed of s "blocks" with n entries each.

v is an eigenvector of $A(G_s)$ associated to the eigenvalue $-d_1$.

Switching the blocks (1,0,. . . ,0) give us $s-1$ linearly independent eigenvectors of $-d_1$.

For $-d_2$ consider the blocks (0,-1,. . . ,0) and (0,1,. . . ,0). The rest follows similarly. □

Corollary

If G is a connected graph then $A(G_s)$ has at least $n \cdot (s-1)$ negative eigenvalues and hence, at most n non negative eigenvalues.

Let X be a $m \times n$ matrix and let Y be a $p \times q$ matrix. The **kronecker product** $X \otimes Y$ is the $mp \times nq$ matrix:

$$X \otimes Y = \begin{bmatrix} x_{11}Y & \dots & x_{1n}Y \\ \vdots & \ddots & \vdots \\ x_{m1}Y & \dots & x_{mn}Y \end{bmatrix}.$$

Proposition

$$A(G_s) = (J_s \otimes Q(G)) + (I_s \otimes -D(G)),$$

where J_s is the $s \times s$ matrix with 1 on all entries and I_s is the $s \times s$ identity matrix.

Theorem A: Let X be a $n \times n$ matrix and Y a $m \times m$ matrix. If $x_1 \geq \ldots \geq x_n$ are the eigenvalues of X and $y_1 \geq \ldots \geq y_m$ the eigenvalues of Y, then the nm eigenvalues of $X \otimes Y$ are: $x_1 y_1, \ldots, x_1 y_m, x_2 y_1, \ldots, x_2 y_m, \ldots, x_n y_1, \ldots, x_n y_m$.

Theorem B: Let X and Y be square $n \times n$ Hermitian matrices with eigenvalues $x_1 \geq \ldots \geq x_n$ and $y_1 \geq \ldots \geq y_n$ respectively. If the eigenvalues of the sum $Z = X + Y$ are $z_1 \geq \ldots \geq z_n$, then $x_k + y_n \leq z_k \leq x_k + y_1$.

- All previously defined matrices are real and symmetric, so they are Hermitian (a square matrix that is equal to its own conjugate transpose).

Proposition

If G be a graph with n vertices, then

$$s.q_1(G) - \Delta(G) \leq \lambda_1(G_s) \leq s.q_1(G) - \delta(G).$$

Proof: For the left inequality, we observe that the largest eigenvalue of J_s is s. Thus, by Theorem A, the largest eigenvalue of $J_s \otimes Q(G)$ is $s \cdot q_1(G)$.
Also, the smallest eigenvalue of $-D(G)$ is $-\Delta(G)$.
By Theorem A, the smallest eigenvalue of $I_s \otimes -D(G)$ is $-\Delta(G)$.
Since $A(G_s) = (J_s \otimes Q(G)) + (I_s \otimes -D(G))$, from Theorem B, we have that $s \cdot q_1(G) - \Delta(G) \leq \lambda_1(G_s)$.

For the right inequality, again, the largest eigenvalue of $J_s \otimes Q(G)$ is $s.q_1(G)$ and now we observe that the largest eigenvalue of $I_s \otimes -D(G)$ is $-\delta(G)$.
From Theorem B $\lambda_1(G_s) \leq s.q_1(G) - \delta(G)$.

Relations between spectral and hypergraph parameters

Proposition: Given a connected graph G the number of distinct eigenvalues of A(G) is at least d(G)+1.

This result is still true on hypergraphs with a very similar proof.

Lemma

Let H be a hypergraph and $A = A(H)$ its adjacency matrix. $(A^l)_{i,j} > 0$ if there is a path with length l connecting two distinct vertices i and j, and $(A^l)_{i,j} = 0$ otherwise (where $(A^l)_{i,j}$ denotes the entry i, j of $A(H)^l$).

Proposition

If H is a connected hypergraph then
$|\{\text{distinct eigenvalues of } A(H)\}| \geq d(H) + 1.$

Proof: Let $\lambda_1, \ldots, \lambda_t$ be all the distinct eigenvalues of $A = A(H)$. Then $(A - \lambda_1 I) \ldots (A - \lambda_t I) = 0$. So, we have that A^t is a linear combination of $A^{t-1}, \ldots, A, I$.

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Suppose by contradiction that $t \leq d(H)$.
Hence $\exists i, j$ such that $d(i,j) = t$.
From our previous lemma, we have that $(A^t)_{i,j} > 0$, and since there exists no path with length shorter than t joining i and j, $(A^{t-1})_{i,j} = 0, \ldots, (A)_{i,j} = 0, (I)_{i,j} = 0$.
This is a contradiction, since
$(A^t)_{i,j} = c_1(A^{t-1})_{i,j} + \ldots + c_{t-1}(A)_{i,j} + c_t(I)_{i,j}$. □

Corollary

If G is connected then $|\{\textit{distinct eigenvalues of } A(G_s^k)\}| \geq d(G)+1.$

In other words, to find connected hypergraphs of the class G_s^k with few distinct adjacency eigenvalues, we have to look for graphs G with small diameter.

Proposition: If G is a graph, then $\alpha(G) \leq min\{\lambda(G)^{-}, \lambda(G)^{+}\}$, where $\lambda(G)^{-}$ is the number of non positive eigenvalues of $A(G)$ and $\lambda(G)^{+}$ is the number of non negative eigenvalues of $A(G)$.

This is not valid for hypergraphs in general and it is never true for connected hypergraphs of the class G_s.

Proposition

If G is a connected graph on n vertices, then $\alpha(G_s) > min\{\lambda(G_s)^{-}, \lambda(G_s)^{+}\}$.

Proof: From previous Corollary, we have that $A(G_s)$ has at most n non negative eigenvalues. Hence, from the independence number proposition: $\alpha(G_s) \geq (s-1)n + \alpha(G) > n \geq \lambda(G_s)^{+} \geq min\{\lambda(G_s)^{-}, \lambda(G_s)^{+}\}$. □

Proposition: For any graph G, $\frac{|V(G)|}{\alpha(G)} \leq \lambda_1(G) + 1$.

This fact has not yet been generalized for hypergraphs and we prove its validity for connected hypergraphs in the class G_s.

Proposition

If G is connected on n vertices then $\frac{|V(G_s)|}{\alpha(G_s)} \leq \lambda_1(G_s) + 1$.

Proof: From the independence number proposition, we have that $\frac{|V(G_s)|}{\alpha(G_s)} = \frac{sn}{\alpha(G_s)} \leq \frac{sn}{(s-1)n+\alpha(G)} \leq \frac{sn}{(s-1)n} = \frac{s}{s-1}$.

We have that $s \cdot q_1(G) - \Delta(G) \leq \lambda_1(G_s)$,(bound on $\lambda_1(G_s)$).

Thus, it suffices to show that $\frac{s}{(s-1)} \leq s.q_1(G) - \Delta(G) + 1$ or, in other words, that $s \leq (s-1)(s.q_1(G) - \Delta(G) + 1)$.

Since $s > 1$, if $s.q_1(G) - \Delta(G) + 1 \geq 2$ then the above inequality is valid, indeed: $s.q_1(G) - \Delta(G) + 1 \geq$
$s(\Delta(G) + 1) - \Delta(G) + 1 = (s-1)\Delta(G) + s + 1 \geq 2$.

Note that the first inequality holds because: If G is a connected graph then $q_1(G) \geq \Delta(G) + 1$. □

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